Sub-Context-Sensitive Groups

22nd Postgraduate Group Theory Conference

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The Word Problem

Fix some group presentation $\langle X \mid R \rangle$ for G with canonical projection π.

Word Problem for G

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Question: Does $\pi(x) = \pi(y)$?

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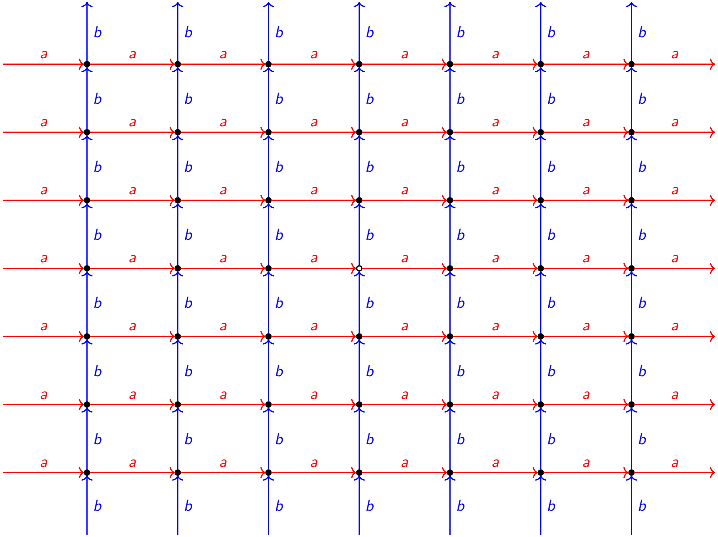
Word problem for groups dates back to Dehn (1911).

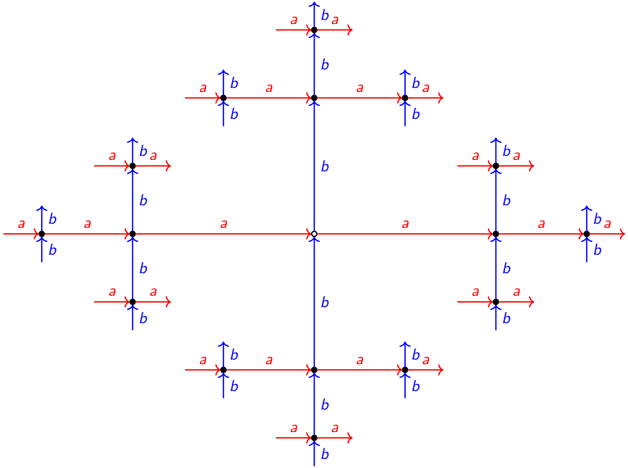
Theorem (Novikov (1955))

There is a finite group presentation with an undecidable word problem.

Cayley Graph of $\mathbb{Z}^2$



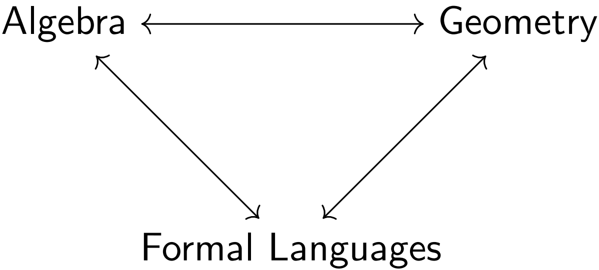
Cayley Graph of F_2



Geometric Group Theory

Algebra $\longleftrightarrow$ Geometry

Geometric Group Theory



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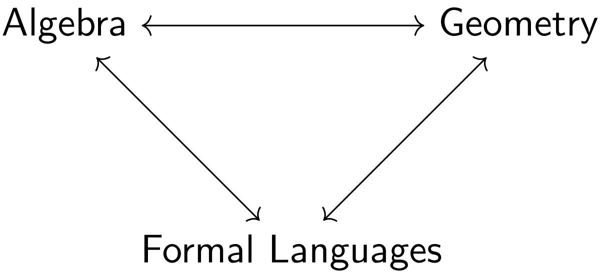

NB All groups and presentations in this talk will be finitely generated.

Chomsky Hierarchy

Definition (Family of Languages)

A language $L \subseteq A^*$ is simply a set of strings over some alphabet A. A family of languages is simply a class of languages which is closed under isomorphism (renaming of symbols). That is, alphabets don't matter.

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Class/Device	Grammar	Machine
$\mathcal{RAT}$	Regular	Finite State Automata
$\mathcal{CF}$	Context-Free	Pushdown Automata
$\mathcal{CS}$	Monotonic	Linear Bounded Automata
$\mathcal{RE}$	Unrestricted	Turing Machines

Group Properties I

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Suppose $\mathcal{F}$ is some family of languages, and $\mathcal{P}$ is some property of a presentation. What we want is for the following statement:

A group G has property $\mathcal{P}$ if there exists a presentation $\langle X \mid R \rangle$ of G such that $\mathrm{WP}_X(G) \in \mathcal{F}$.

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> A group G has property $\mathcal{P}$ if there exists a presentation $\langle X \mid R \rangle$ of G such that $\mathrm{WP}_X(G) \in \mathcal{F}$.

to be equivalent to:

> A group G has property $\mathcal{P}$ if for all presentations $\langle X \mid R \rangle$ of G, $\mathrm{WP}_X(G) \in \mathcal{F}$.

Group Properties II

Definition

A family of languages $\mathcal{F}$ is said to be closed under inverse homomorphisms if for all alphabets A, B, all languages $L \subseteq A^*$ in $\mathcal{F}$, all free monoid homomorphisms $\varphi : B^* \to A^*$, the preimage $\varphi^{-1}(L)$ is in $\mathcal{F}$.

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Theorem

Let G be a group and $\mathcal{F}$ be a family of languages which is closed under inverse homomorphisms. If $\mathrm{WP}_X(G) \in \mathcal{F}$ for some presentation $\langle X \mid R \rangle$ of G, then it is for all presentations of G.

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This is exactly what we wanted. If we have a class of groups defined by some (algebraic) property $\mathcal{P}$, and a language family $\mathcal{F}$ closed under inverse homomorphisms, deciding if all the word problems of the groups in the class amounts to only checking if there is a presentation of each group with word problem in the family.

Grammars

Definition (String Rewriting)

Given an alphabet Σ and a set of rewriting rules $\mathcal{R} \subseteq \Sigma \times \Sigma^*$, we define the binary relation $\to_{\mathcal{R}}$ on Σ^* by $x \to_{\mathcal{R}} y$ iff $x = ulv$ and $y = urv$ for some $u, v \in \Sigma^*$ and $(l, r) \in \mathcal{R}$.

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Definition (String Grammar)

A grammar is a $\mathcal{G} = (\Sigma, A, S, \mathcal{R})$ where Σ is some alphabet, $A \subseteq \Sigma$ are the terminals, $S \in \Sigma^*$ is the start string, and $\mathcal{R} \subseteq \Sigma \times \Sigma^*$. The language generated by $\mathcal{G}$ is $\mathrm{L}(\mathcal{G}) = \{w \in A^* \mid S \to_{\mathcal{R}}^* w\}$.

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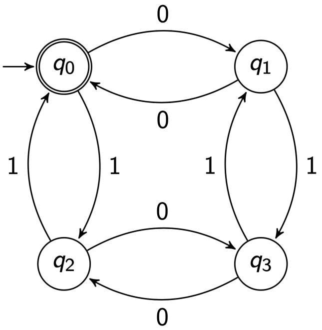
Call $\mathcal{G}$ regular if the rules are of form $X \to aY$ ($X, Y \in \Sigma \setminus A$, $a \in A$).

Call $\mathcal{G}$ context-free if the rules of form $X \to w$ ($X \in \Sigma \setminus A$, $w \in w^*$).

Call $\mathcal{G}$ monotone if the rules are are length non-decreasing.

Machines

The most basic type of machine is a finite state automaton.



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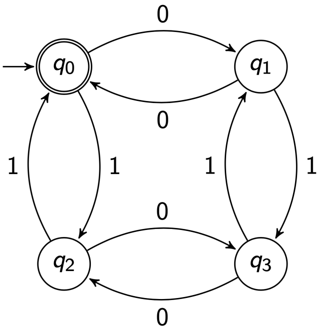

Pushdown automata have access to a stack, and Turing machines to an infinite tape. Linear bounded automata are Turing machines with a linearly (wrt the input string length) bounded tape.

Regular Groups

Call a group regular if it has regular word problem.

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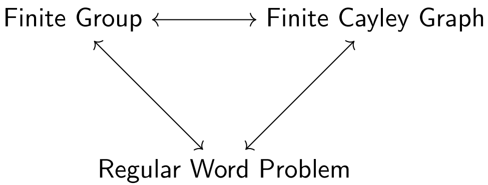

Proof

Proof: For the reverse direction, we can simply set the generating set to be all the elements in the group, and then have an FSA simply evaluate the finite multiplication table for the group, by having a state for each group element and having a transition between elements whenever the state multiplied by the edge label equals the label of the target state. Make the identity-labelled state the initial and final state.

For the forward direction, suppose by contradiction that the group is infinite. Then there must be arbitrarily long string such that no non-empty substring equals the identity in the group, since if there were a bound on the length of such words, then every element of the group would be represented by only finitely many strings... a contradiction. Now, let $\mathcal{A}$ be an FSA for the word problem, with n states, and choose a string w of length at least $n+1$ such that no non-empty substring is equal to the identity in the group. The, machine must visit a q state twice. Say, if visits q after reading u and again after then reading v, then $w = uvy$. Now, if $uu^{-1} = 1$ but $uvu^{-1} \neq 1$, $\mathcal{A}$ must accept or reject both strings... a contradiction! □

Context-Free Groups

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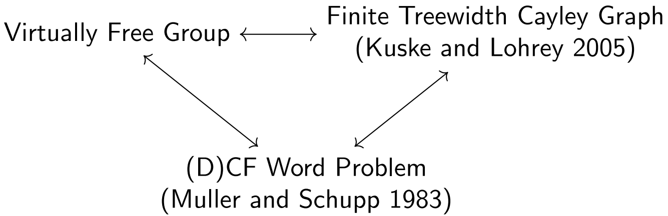
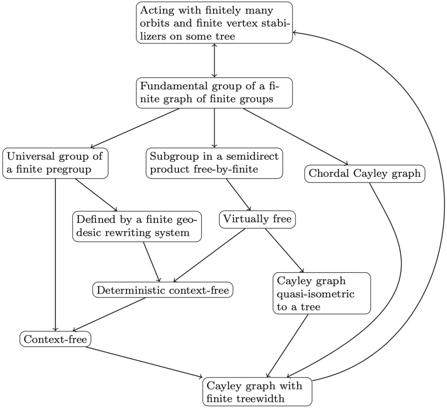


Proof

Diekert and Weiß (2017) provide an as simple as possible re-proof of this result (and other characterisations).



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The real-time languages are a sub-family of the DCS languages.

Theorem (Holt and Rees (2001))

Groups with generalised Dehn algorithms are real-time.

Theorem (Goodman and Shapiro (2008))

Nilpotent groups and geometrically finite hyperbolic groups have generalised Dehn algorithms.

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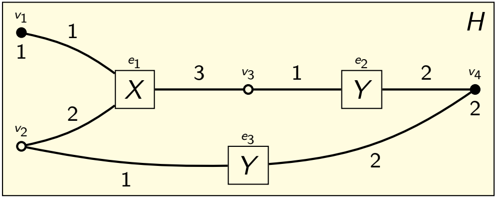
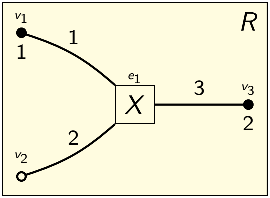
Theorem (Engelfriet and Heyker (1991) and Weir (1992))

The MCF languages are exactly the string languages generated by hyperedge replacement grammars (see, for example, Drewes, Kreowski, and Habel (1997)).

Hyperedge Replacement





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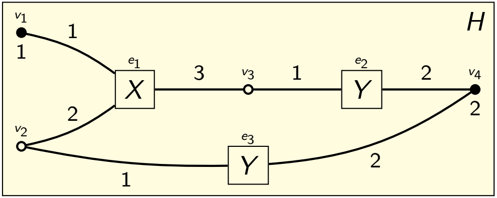

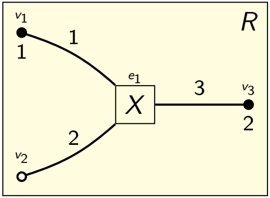

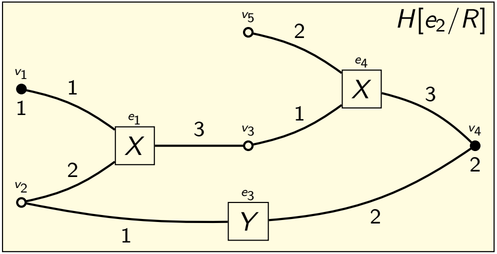

Lindenmayer Systems

There lots of other well-behaved language classes sitting in between the CF and CS classes, such as the indexed languages (Aho 1968) and their subclass of ET0L languages (Rozenberg and Salomaa 1980).

It is not known if there are any groups with indexed word problems other than the virtually free groups.

In particular, we don't know if any hyperbolic groups (other than the virtually free groups) have ET0L word problems (Ciobanu, Elder, and Ferov 2018), such as the fundamental group of the double torus.

Maybe we can find more groups if we mix together the ideals from ET0L and MCF. I have done this, constructing a new language family which has lots of closure properties (Campbell 2021). I am yet to find a word problem in this class though (other than a virtually free one)!

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