Parallel Hyperedge Replacement String Languages

11th International Workshop on Computing with Terms and Graphs

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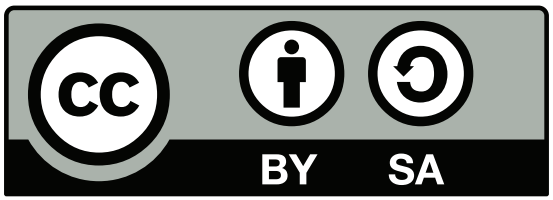
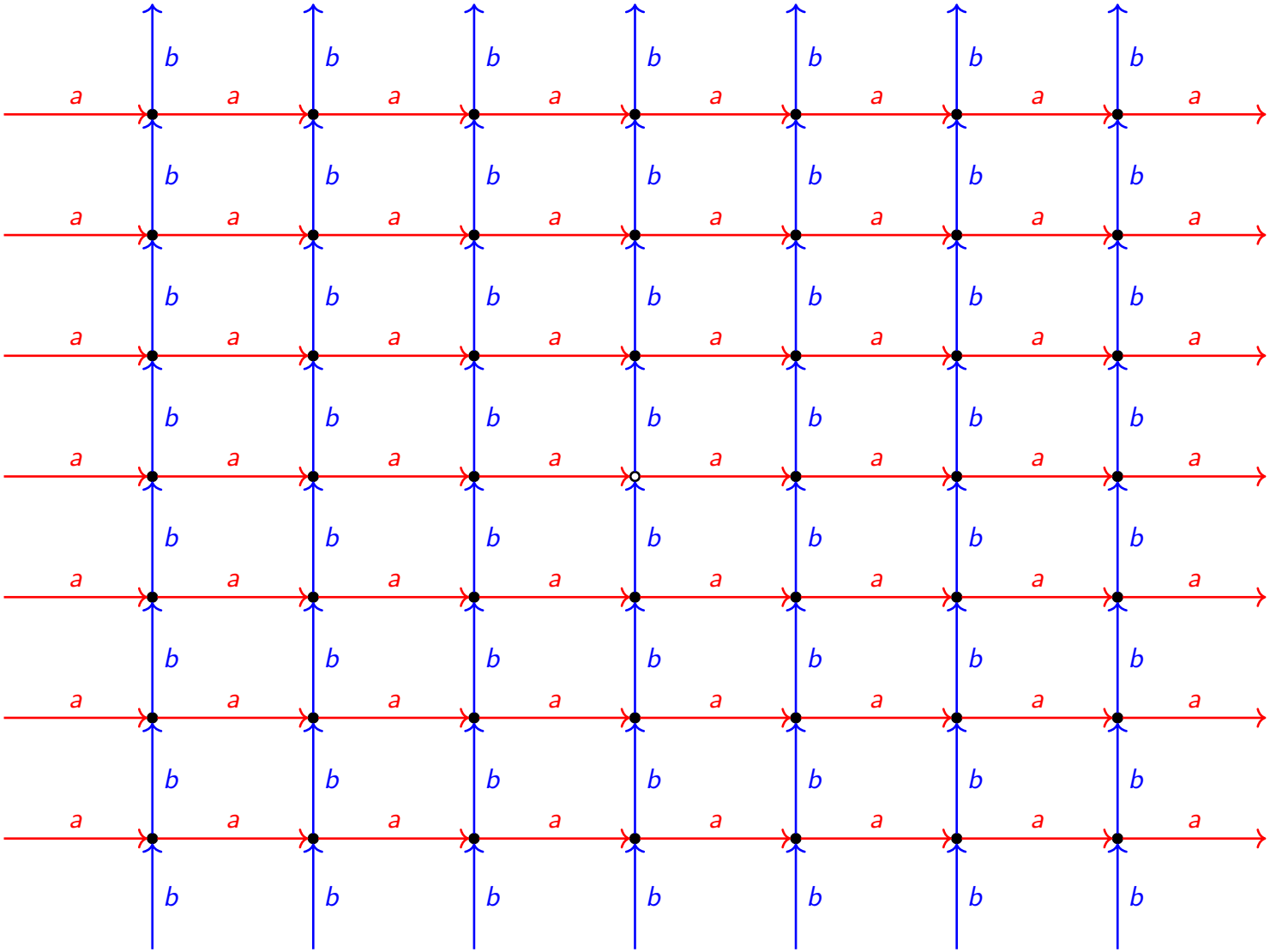
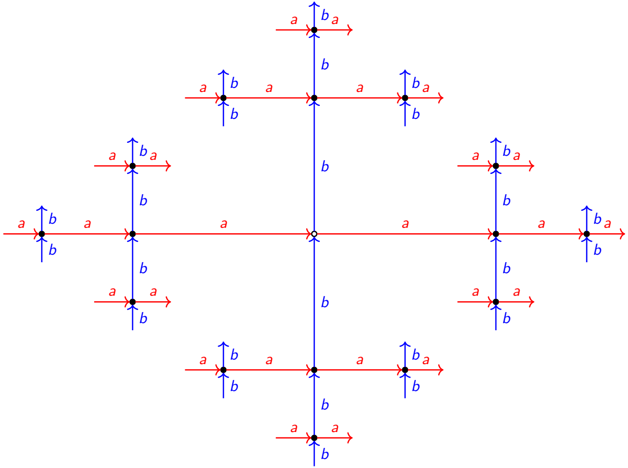


Cayley Graph of $\mathbb{Z}^2$



Cayley Graph of F_2



The Word Problem

Definition (Word Problem)

Given a group presentation $\langle X \mid R \rangle$ for a group G, the word problem is the membership problem for the string language:

$$\mathrm{WP}_X(G) = \{w \in (X \cup X^{-1})^* \mid w =_G 1_G\}.$$

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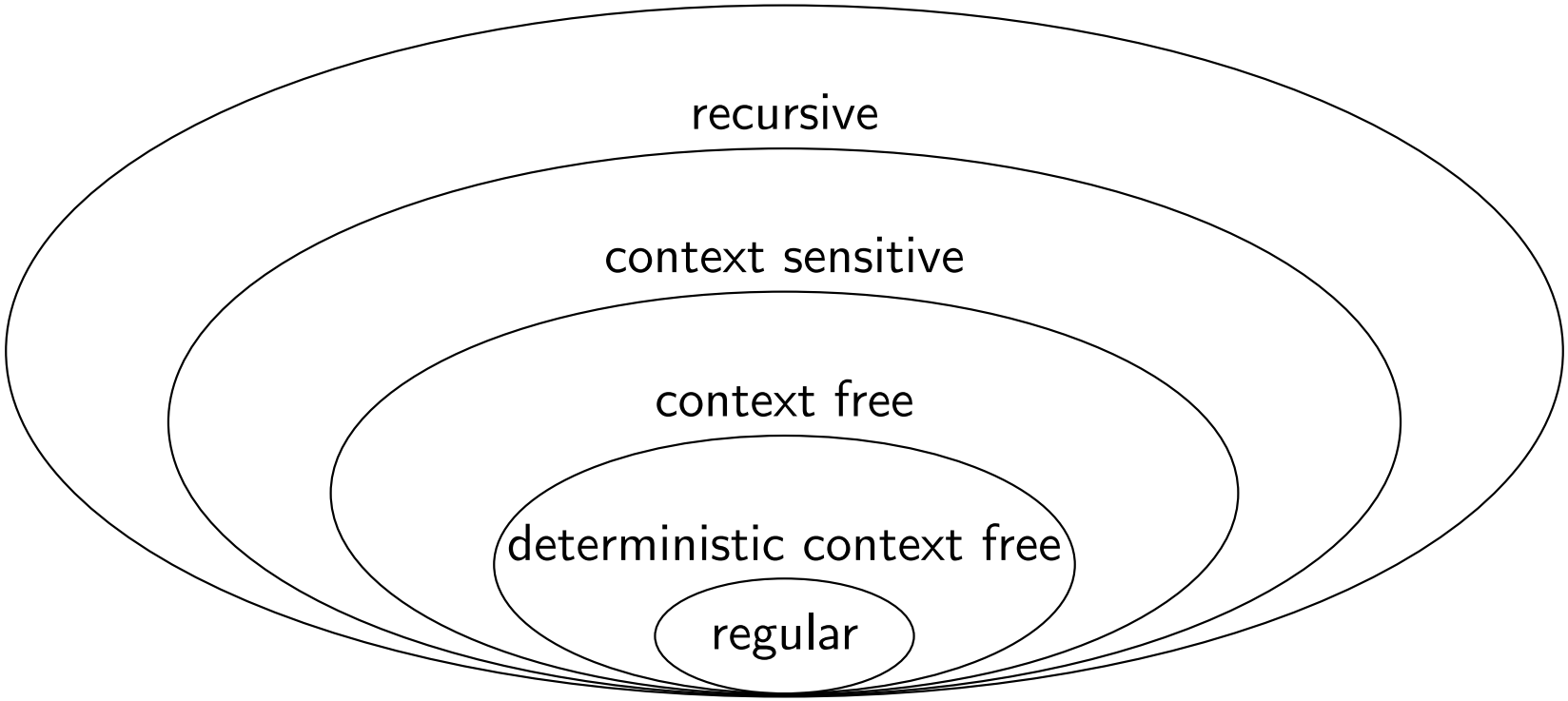
Answer: Maybe...

Context-Free Groups



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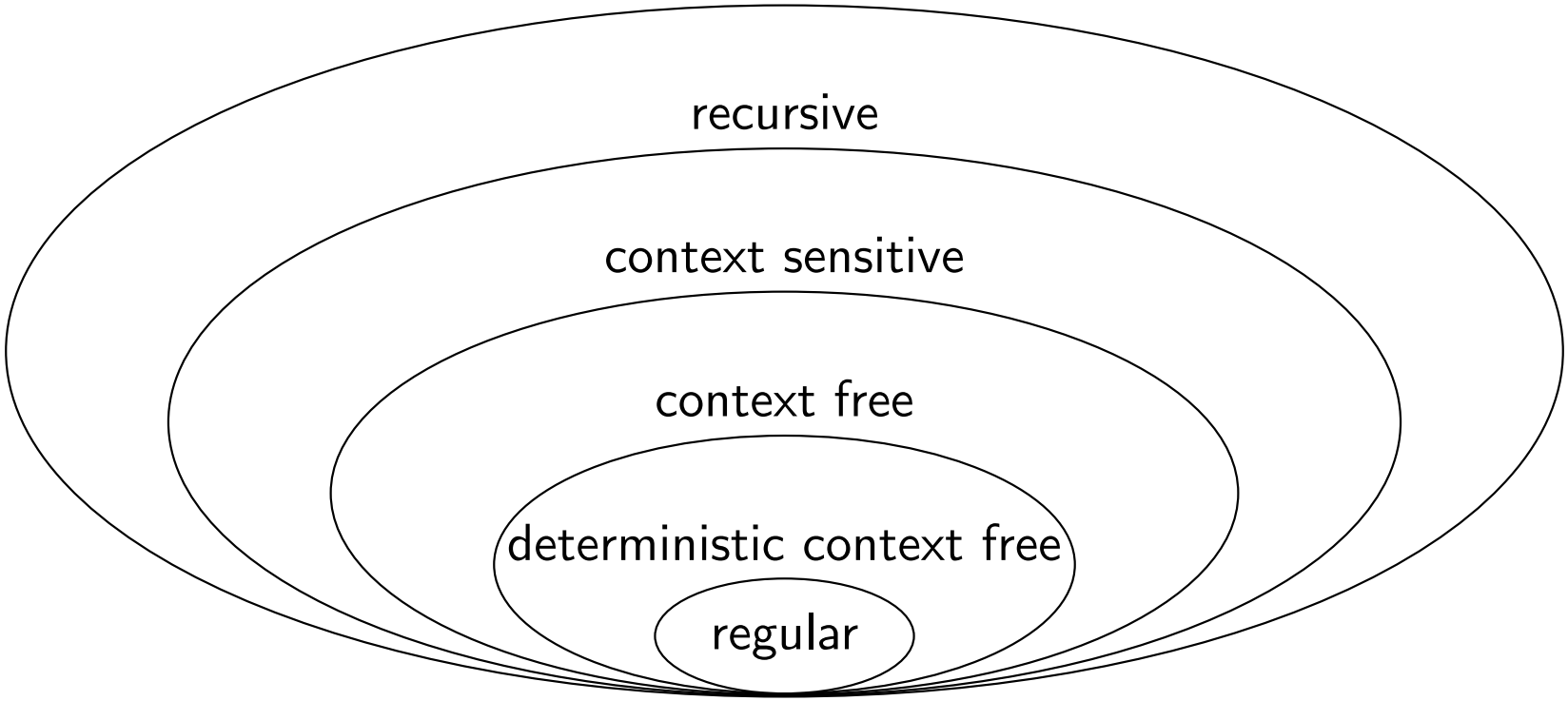

Theorem (Anisimov (1971) and Muller and Schupp (1983))

1. A presentation defines a finite group iff it has regular WP;
2. A presentation defines a virtually free group iff it has (D)CF WP.

Multiple Context-Free

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Theorem (Engelfriet and Heyker (1991) and Weir (1992))

The MCF languages are exactly the string languages generated by HR grammars (Drewes, Kreowski, and Habel 1997).

Lindenmayer Systems

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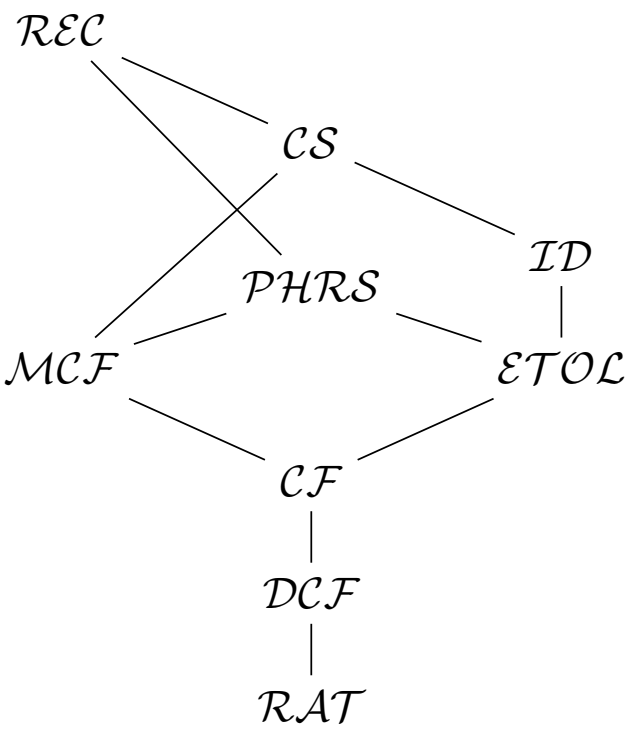
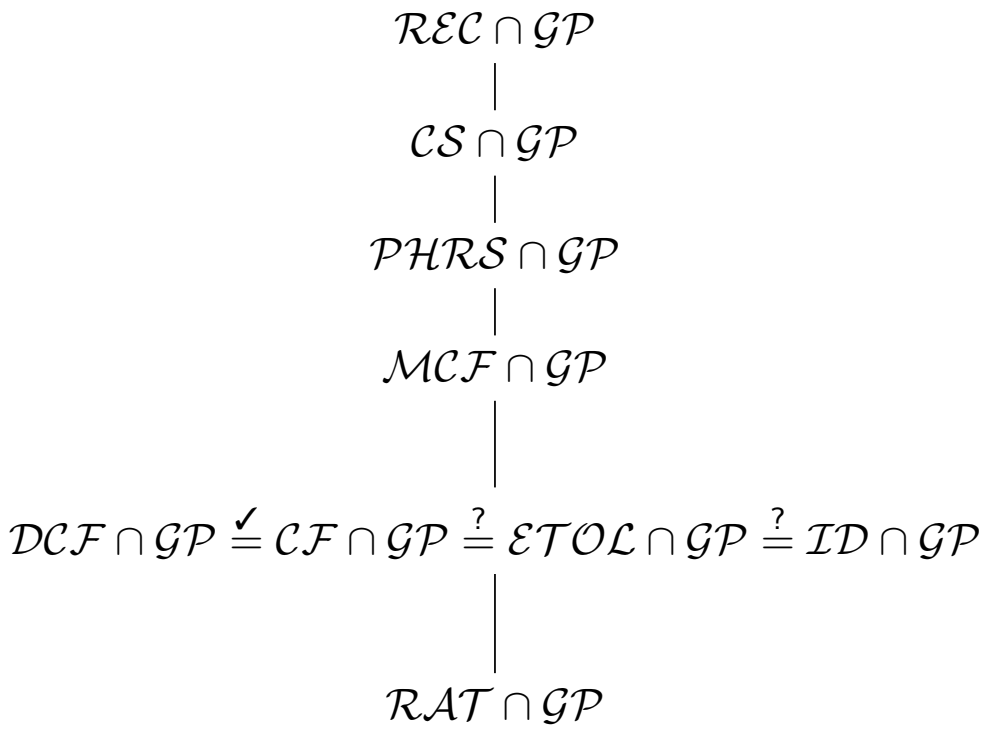
What if we tried to mix together ideas from MCF and ET0L. . . parallel hyperedge replacement! Do the word problems of hyperbolic groups lie within this class? I should mention forms of parallel HR have been considered before (Habel 1992; Kreowski 1993), the work is not extensive and does not consider rational control or string generational power.

Taking Stock

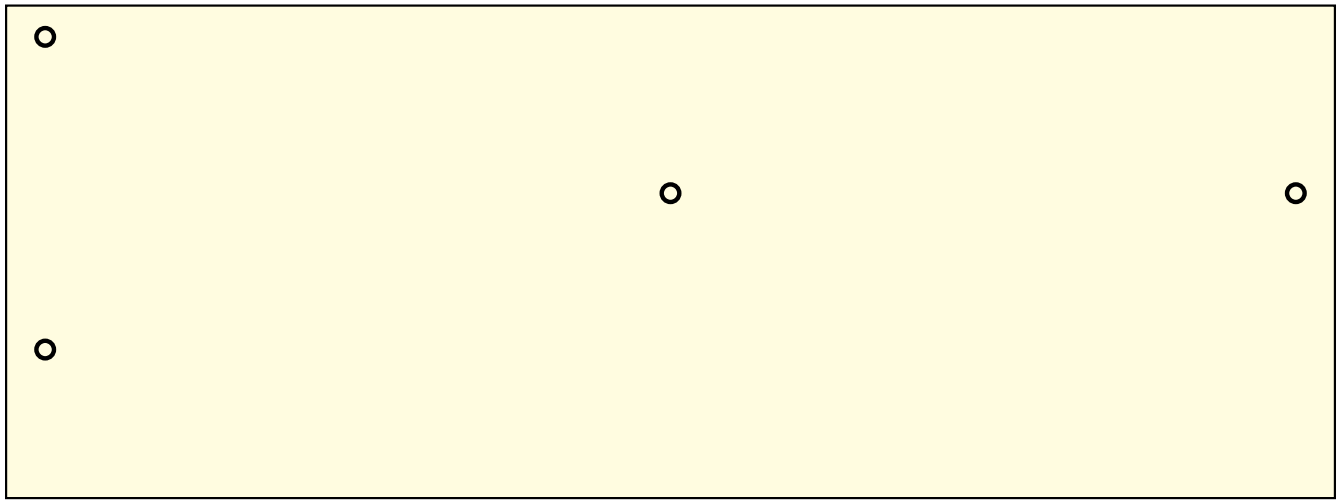


(a) Proved String Language Hierarchy



(b) Conjectured Group Language Hierarchy

Hypergraphs I



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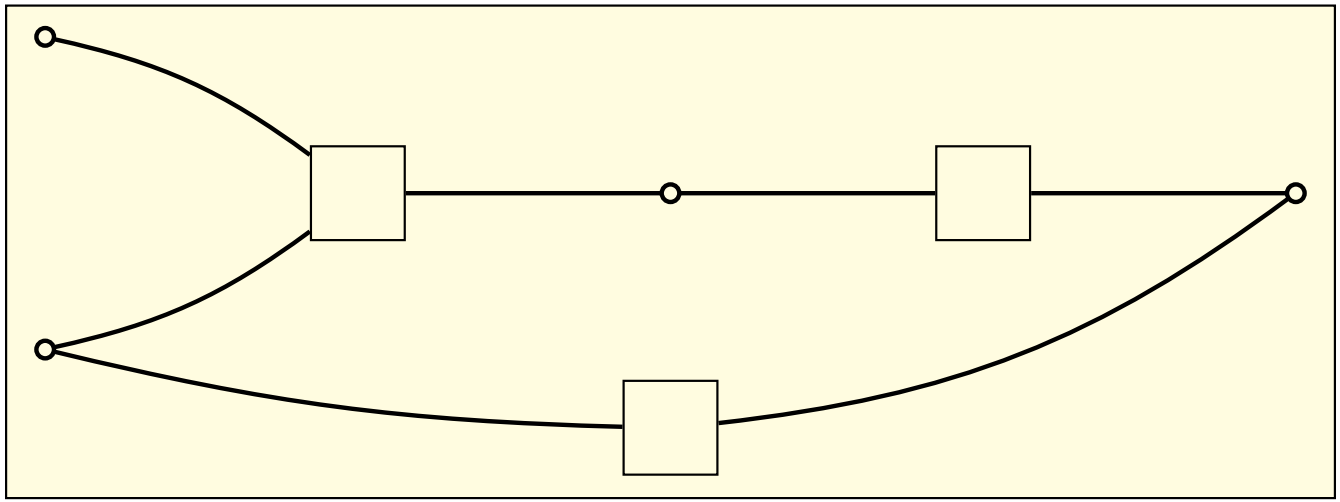

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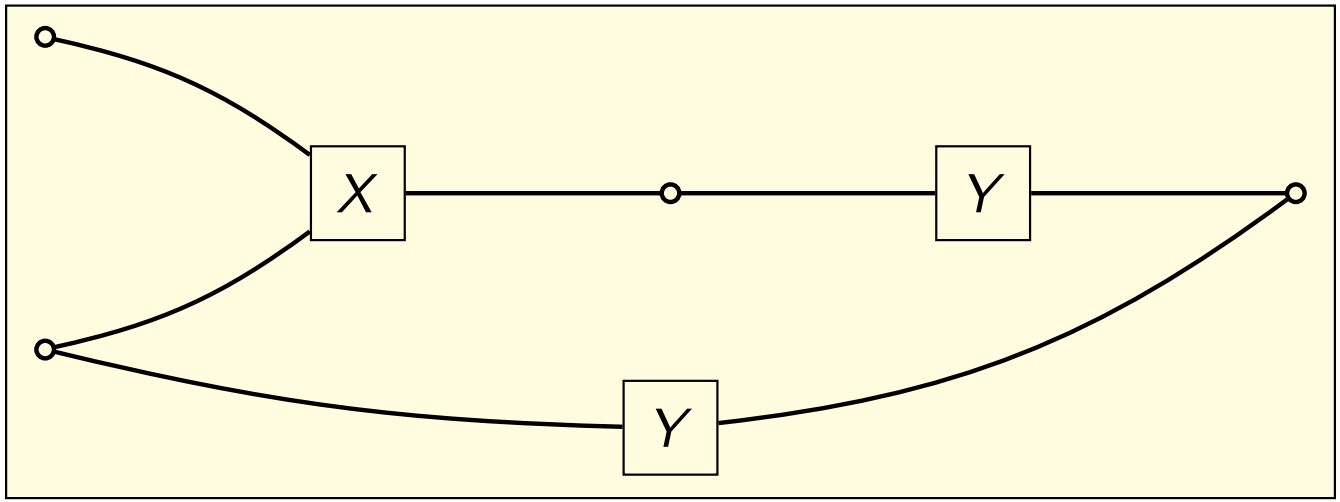

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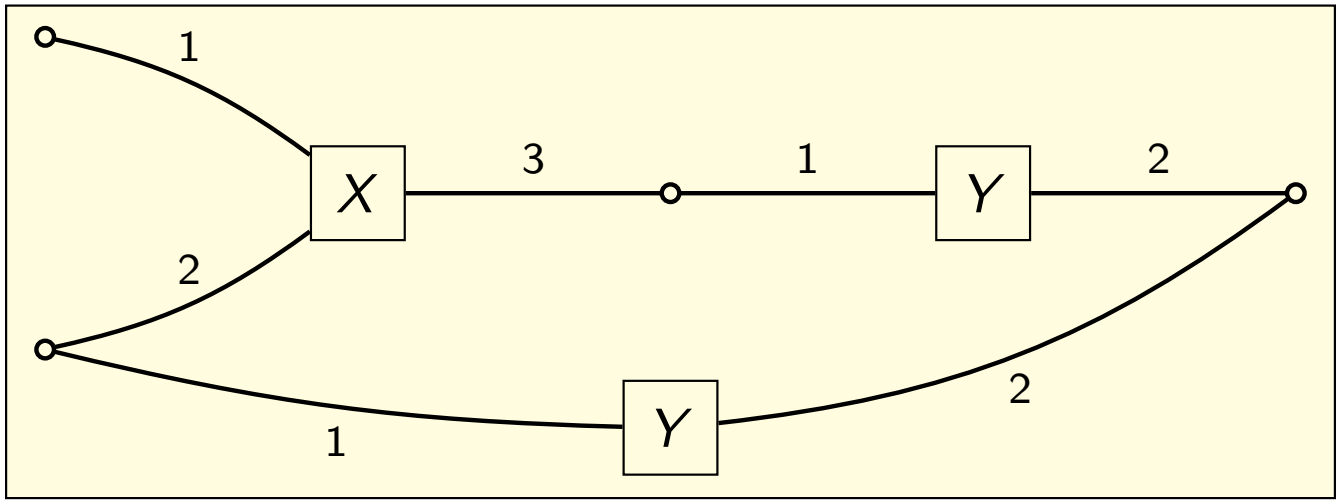

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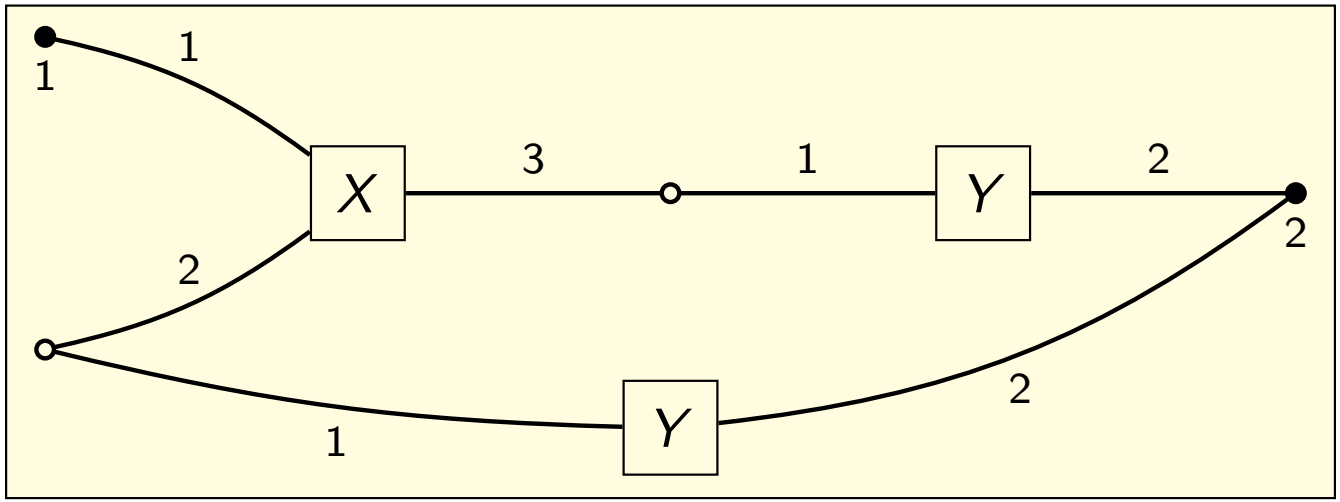

Hypergraphs II

Definition (Signature)

A signature is a pair $\mathcal{C} = (\Sigma, \mathrm{type})$ where Σ is some finite label set, and $\mathrm{type} : \Sigma \to \mathbb{N}$ is a typing function which assigns to each label an arity.

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A hypergraph over $\mathcal{C}$ is a tuple $H = (V_H, E_H, \mathrm{att}_H, \mathrm{lab}_H, \mathrm{ext}_H)$ where:

1. V_H is a finite set of nodes;
2. E_H is a finite set of hyperedges;
3. $\mathrm{att}_H : E_H \to \mathrm{seq}(V_H)$ is the attachment function;
4. $\mathrm{lab}_H : E_H \to \Sigma$ is the labelling function;
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such that labelling is compatible with typing ($\mathrm{type} \circ \mathrm{lab}_H = |\cdot| \circ \mathrm{att}_H$).

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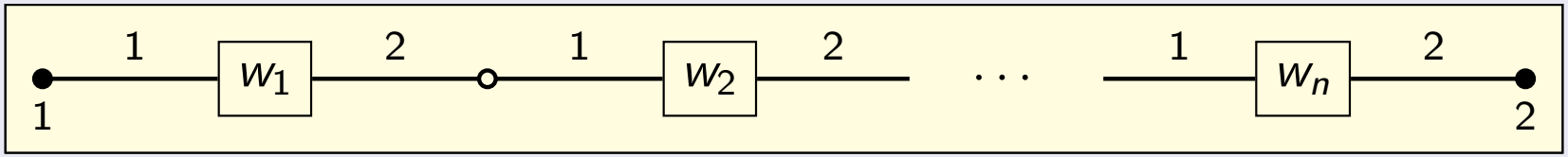
The class of all hypergraphs over $\mathcal{C}$ is denoted $\mathcal{H}_{\mathcal{C}}$.

Strings and Handles

Definition (String Graph)

Given a non-empty word $w = w_1 w_2 \cdots w_n$, we define its string graph $w^\bullet$:



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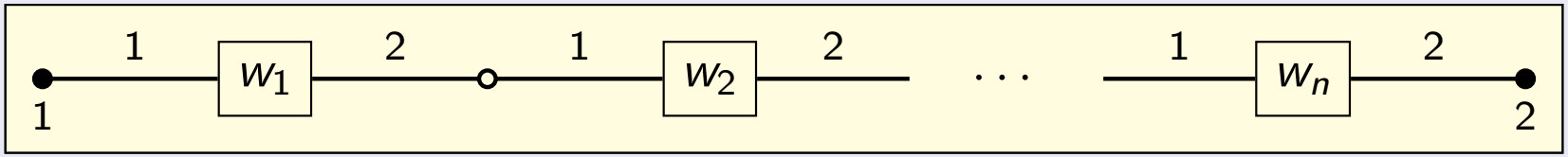
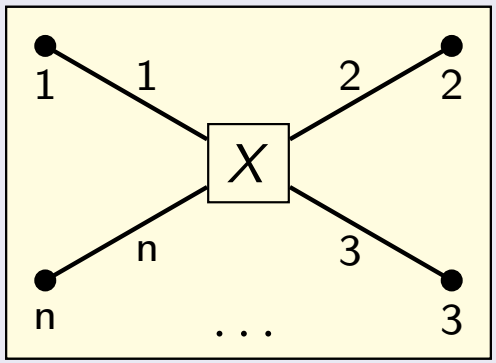
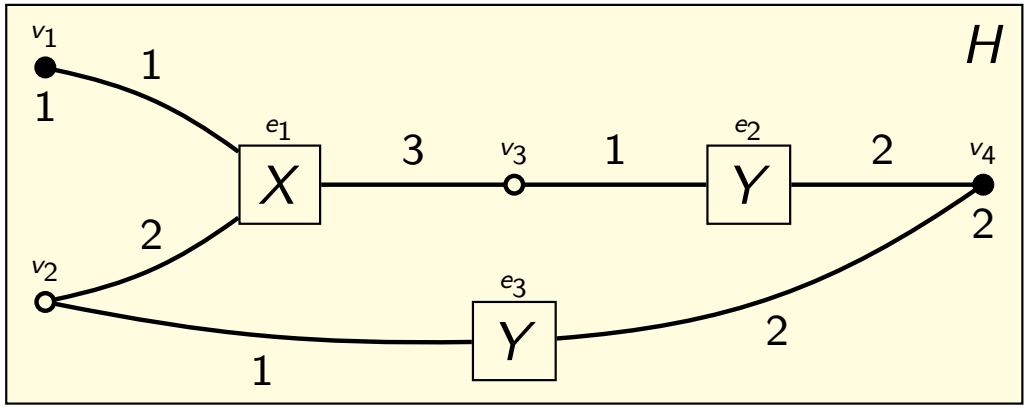


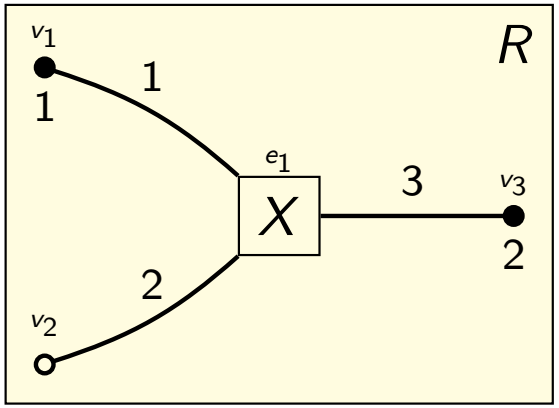
Definition (Handle)

Given a label X of type n, we define its handle $X^\bullet$:



Replacement





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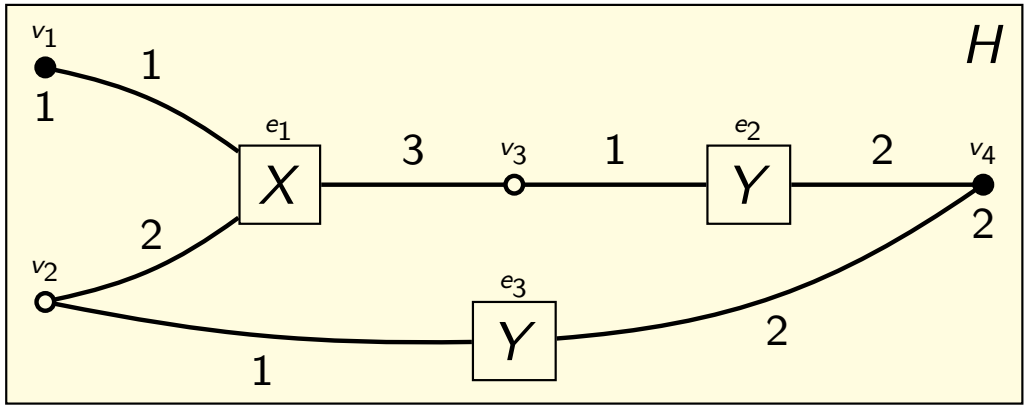

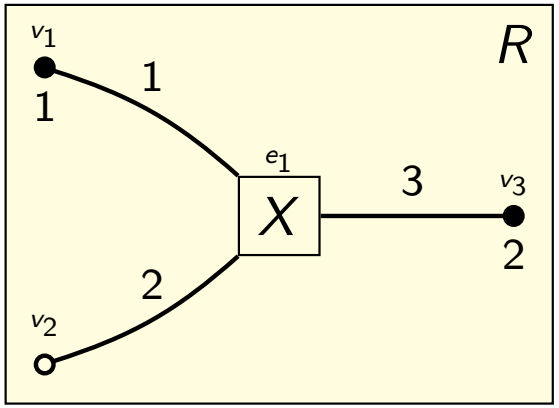

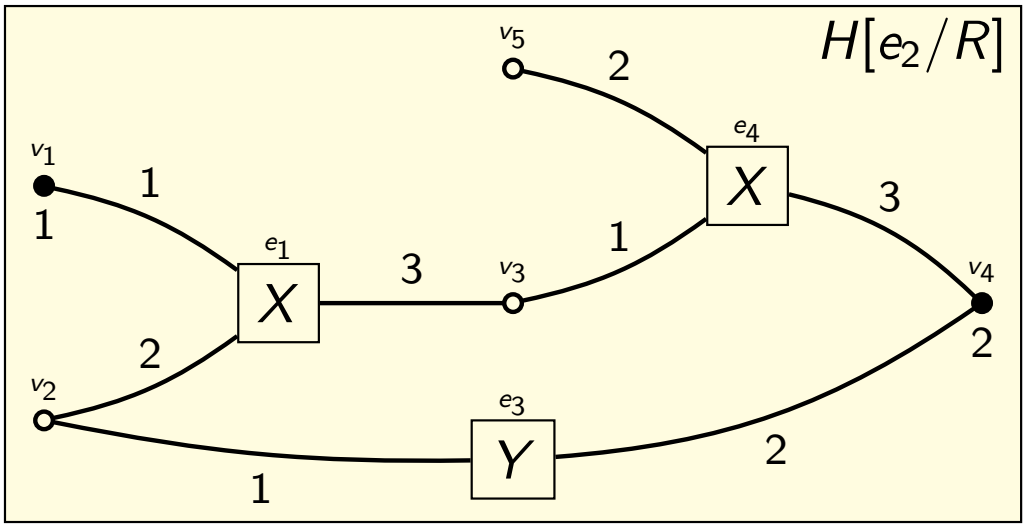

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Given $H \in \mathcal{H}_{\mathcal{C}}$ and $\mathcal{R}$ a set of rules, if $e \in E_H$ and $(\mathrm{lab}_H(e), R) \in \mathcal{R}$, then we say that H directly derives $H' \cong H[e/R]$, and write $H \Rightarrow_{\mathcal{R}} H'$.

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Definition (Derivation)

H derives H' if there is a sequence $H \Rightarrow_{\mathcal{R}} H_1 \Rightarrow_{\mathcal{R}} \cdots \Rightarrow_{\mathcal{R}} H_k \cong H'$ for some $k \in \mathbb{N}$. We write $H \Rightarrow^k_{\mathcal{R}} H'$ or $H \Rightarrow^*_{\mathcal{R}} H'$.

HR Grammars

Definition (HR Grammar)

A HR grammar of order k is a system $\mathcal{G} = (\mathcal{C}, N, S, \mathcal{R})$ where:

1. $\mathcal{C} = (\Sigma, \mathrm{type})$ is a signature;
2. $N \subseteq \Sigma$ is the set of non-terminal labels;
3. $S \in N$ is the start symbol;
4. $\mathcal{R}$ is a finite set of rules over N;

with $\max(\{\mathrm{type}(R) \mid (L, R) \in \mathcal{R}\}) \leq k$. The generated language is:

$$\mathrm{L}(\mathcal{G}) = \{H \in \mathcal{H}_{\mathcal{C}} \mid S^{\bullet} \Rightarrow^{*}_{\mathcal{R}} H \text{ with } \mathrm{lab}_{H}^{-1}(N) = \emptyset\} \subseteq \mathcal{H}_{\mathcal{C}}.$$

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The class of HR languages is the union of all k-HR languages for $k \in \mathbb{N}$.

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Theorem (Precise Theorem 5)

For all $k \geq 1$, $\mathcal{HRS}_{2k} = \mathcal{HRS}_{2k+1} = \mathcal{MCF}_k$.

Parallel Derivations

Definition (Parallel Direct Derivation)

Let $H \in \mathcal{H}_{\mathcal{C}}$ with $E_H = \{e_1, \ldots e_n\}$, and $\mathcal{R}$ be a set of rules. If for each $e_i \in E_H$, there is an $R_i \in \mathcal{H}_{\mathcal{C}}$ such that $(\mathrm{lab}_H(e_i), R_i) \in \mathcal{R}$, then H parallelly directly derives $H' \cong H[e_1/R_1, \ldots, e_n/R_n]$, and write $H \Rrightarrow_{\mathcal{R}} H'$.

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Definition (Parallel Derivation)

Let $\mathcal{S} = \{\mathcal{R}_i \mid i \in I\}$ be a finite set of rule sets indexed by I, and $\mathcal{M}$ an FSA over I. Then H ($\mathcal{M}$-)parallelly derives H' if there is a sequence $H \Rrightarrow_{\mathcal{R}_{i_1}} H_1 \Rrightarrow_{\mathcal{R}_{i_2}} \cdots \Rrightarrow_{\mathcal{R}_{i_k}} H_k \cong H'$ $(k \in \mathbb{N})$ such that $i_1 i_2 \cdots i_k \in \mathrm{L}(\mathcal{M})$. We write $H \Rrightarrow_{\mathcal{S}}^{\mathcal{M}} H'$, $H \Rrightarrow_{\mathcal{S}}^{i_1 i_2 \cdots i_k} H'$ or $H \Rrightarrow_{\mathcal{S}}^{k} H'$.

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Definition (Table)

A table T is a finite set of rules over Σ such that for each $L \in \Sigma$, there is at least one $R \in \mathcal{H}_\mathcal{C}$ such that $(L, R) \in T$.

PHR Grammars I

Definition (PHR Grammar)

A (k-PHR grammar is a system $\mathcal{G} = (\mathcal{C}, A, S, \mathcal{T}, \mathcal{M})$ where:

1. $\mathcal{C} = (\Sigma, \mathrm{type})$ is a signature;
2. $A \subseteq \Sigma$ is the set of terminal labels;
3. $S \in \Sigma \setminus A$ is the start symbol;
4. $\mathcal{T} = \{T_i \mid i \in I\}$ is a finite set of tables indexed by I;
5. $\mathcal{M} = (Q, I, \delta, i, F)$ is an FSA over I;

with $\max(\{|type|(R) \mid (L, R) \in \bigcup_{T_i \in \mathcal{T}} T_i\}) \leq k$. The generated language is:

$$\mathrm{L}(\mathcal{G}) = \{H \in \mathcal{H}_{\mathcal{C}} \mid S^{\bullet} \Rrightarrow_{\mathcal{T}}^{\mathcal{M}} H \text{ with } \mathrm{lab}_H^{-1}(A) = E_H\} \subseteq \mathcal{H}_{\mathcal{C}}.$$

PHR Grammars I

Definition (PHR Grammar)

A (k-PHR grammar is a system $\mathcal{G} = (\mathcal{C}, A, S, \mathcal{T}, \mathcal{M})$ where:

1. $\mathcal{C} = (\Sigma, \mathrm{type})$ is a signature;
2. $A \subseteq \Sigma$ is the set of terminal labels;
3. $S \in \Sigma \setminus A$ is the start symbol;
4. $\mathcal{T} = \{T_i \mid i \in I\}$ is a finite set of tables indexed by I;
5. $\mathcal{M} = (Q, I, \delta, i, F)$ is an FSA over I;

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The inclusion of rational control, originally considered for ET0L by Asveld 1977, is a convenience to aid with the specification of human understandable grammars. This actually adds no power...

PHR Grammars II

Lemma (Control Removal)

Given a k-PHR grammar $\mathcal{G}$, one can effectively construct a k-PHR grammar $\mathcal{G}'$ without control such that $\mathrm{L}(\mathcal{G}) = \mathrm{L}(\mathcal{G}')$.

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Proof: Encode control in the labels! Make a copy of all the labels for all of the states in the FSA, and moving between control states is synchronized with moving between the copies of labels. □

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Proof: Induction on derivation length. □

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There are 2-PHRS languages that are not semilinear.

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There are 2-PHRS languages that are not semilinear.

Lemma (PHRS Generalises MCF)

For $k \geq 2$, $\mathcal{HRS}_k \subsetneq \mathcal{PHRS}_k$.

Proof: The equivalence of MCF and HRS due to Theorem 15. To see the remainder follows from Theorem 22 and its proof. We get strictness from Proposition 21 together with Lemma 23. □

Closing Conjectures

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Corollary

The word problem of any surface group is a PHRS language.

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