


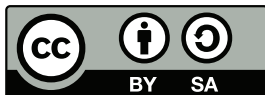
Parallel Hyperedge Replacement String Languages

11th International Workshop on Computing with Terms and Graphs

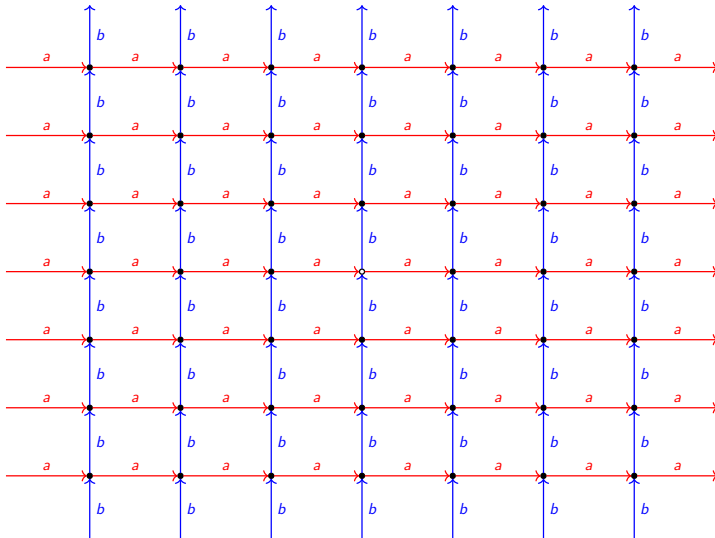
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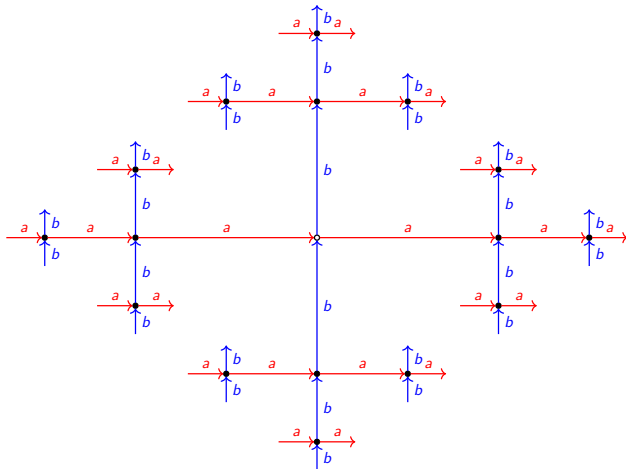
July 2020



Cayley Graph of \mathbb{Z}^2



Cayley Graph of F_2



The Word Problem

Definition (Word Problem)

Given a group presentation $\langle X \mid R \rangle$ for a group G , the word problem is the membership problem for the string language:

$$\text{WP}_X(G) = \{w \in (X \cup X^{-1})^* \mid w =_G 1_G\}.$$

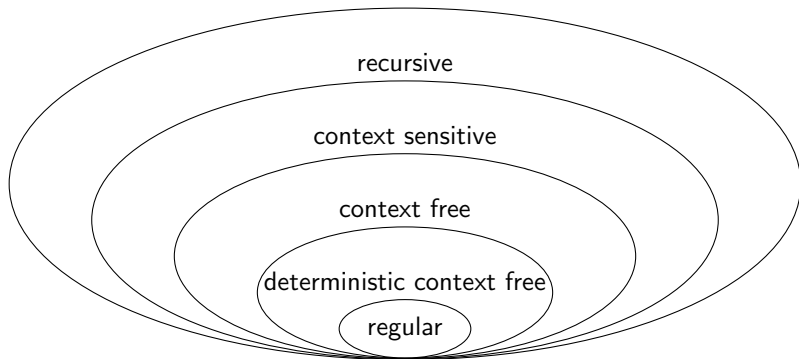
Question: How hard is the word problem?

Answer: It is impossible! There is a finite group presentation with an undecidable word problem (Novikov 1955).

Question: The Chomsky hierarchy is a notion of complexity of a language. Can we characterise the finitely generated groups in terms of language families?

Answer: Maybe...

Context-Free Groups



Theorem (Anisimov (1971) and Muller and Schupp (1983))

- 1 A presentation defines a finite group iff it has regular WP;
- 2 A presentation defines a virtually free group iff it has (D)CF WP.

Multiple Context-Free

Question: how “far away” are interesting families of groups from the context-free languages in the formal language formal language hierarchy?

MCF languages are a conservative extension of the context-free languages, formalised by Seki, Matsumura, Fujii, and Kasami 1991.

Theorem (Ho (2018))

If a presentation defines a virtually Abelian group, then it has MCF WP.

Theorem (Gilman, Kropholler, and Schleimer (2018))

The fundamental group of a hyperbolic three-manifold does not admit a MCF WP.

Theorem (Engelfriet and Heyker (1991) and Weir (1992))

The MCF languages are exactly the string languages generated by HR grammars (Drewes, Kreowski, and Habel 1997).

Lindenmayer Systems

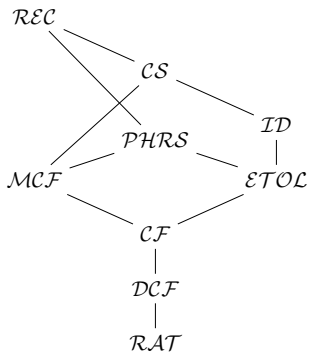
There lots of other well-behaved language classes sitting in between the CF and CS classes, such as the indexed languages (Aho 1968) and their subclass of ETOL languages (Rozenberg and Salomaa 1980).

It is not known if there are any groups with indexed word problems other than the virtually free groups.

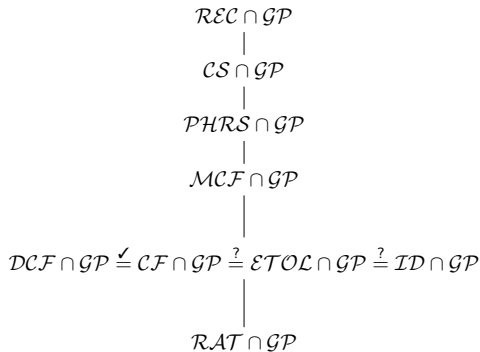
In particular, we don't know if any hyperbolic groups (other than the virtually free groups) have ETOL word problems (Ciobanu, Elder, and Ferov 2018), such as the fundamental group of the double torus.

What if we tried to mix together ideas from MCF and ETOL... parallel hyperedge replacement! Do the word problems of hyperbolic groups lie within this class? I should mention forms of parallel HR have been considered before (Habel 1992; Kreowski 1993), the work is not extensive and does not consider rational control or string generational power.

Taking Stock

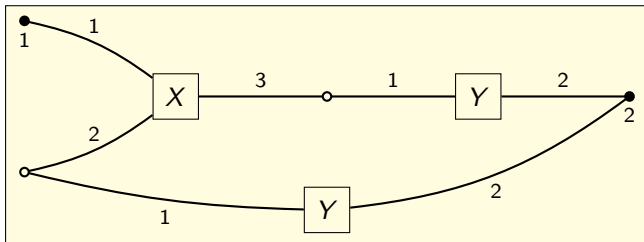


(a) Proved String Language Hierarchy



(b) Conjectured Group Language Hierarchy

Hypergraphs I



Hypergraphs II

Definition (Signature)

A signature is a pair $\mathcal{C} = (\Sigma, \text{type})$ where Σ is some finite label set, and $\text{type} : \Sigma \rightarrow \mathbb{N}$ is a typing function which assigns to each label an arity.

Definition (Hypergraph)

A hypergraph over \mathcal{C} is a tuple $H = (V_H, E_H, \text{att}_H, \text{lab}_H, \text{ext}_H)$ where:

- 1 V_H is a finite set of nodes;
- 2 E_H is a finite set of hyperedges;
- 3 $\text{att}_H : E_H \rightarrow \text{seq}(V_H)$ is the attachment function;
- 4 $\text{lab}_H : E_H \rightarrow \Sigma$ is the labelling function;
- 5 $\text{ext}_H : \text{iseq}(V_H)$ are the external nodes;

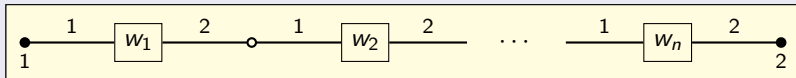
such that labelling is compatible with typing ($\text{type} \circ \text{lab}_H = |\cdot| \circ \text{att}_H$).

The class of all hypergraphs over \mathcal{C} is denoted $\mathcal{H}_{\mathcal{C}}$.

Strings and Handles

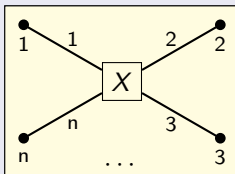
Definition (String Graph)

Given a non-empty word $w = w_1 w_2 \cdots w_n$, we define its string graph w^\bullet :

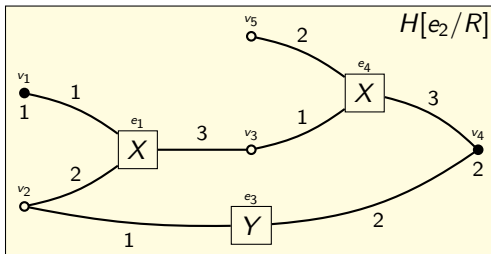
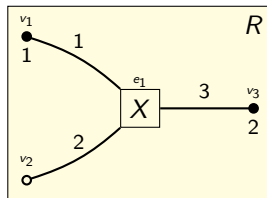
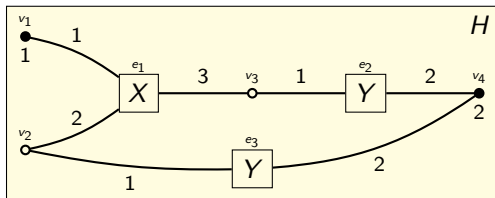


Definition (Handle)

Given a label X of type n , we define its handle X^\bullet :



Replacement



Rules and Derivations

Let $\mathcal{C} = (\Sigma, \text{type})$ be a signature and $N \subseteq \Sigma$ a set of non-terminals.

Definition (Rule)

A rule over N is a pair (L, R) with $L \in N$, $R \in \mathcal{H}_{\mathcal{C}}$, $\text{type}(L) = \text{type}(R)$.

Definition (Direct Derivation)

Given $H \in \mathcal{H}_{\mathcal{C}}$ and \mathcal{R} a set of rules, if $e \in E_H$ and $(\text{lab}_H(e), R) \in \mathcal{R}$, then we say that H directly derives $H' \cong H[e/R]$, and write $H \Rightarrow_{\mathcal{R}} H'$.

Definition (Derivation)

H derives H' if there is a sequence $H \Rightarrow_{\mathcal{R}} H_1 \Rightarrow_{\mathcal{R}} \cdots \Rightarrow_{\mathcal{R}} H_k \cong H'$ for some $k \in \mathbb{N}$. We write $H \Rightarrow_{\mathcal{R}}^k H'$ or $H \Rightarrow_{\mathcal{R}}^* H'$.

HR Grammars

Definition (HR Grammar)

A HR grammar of order k is a system $\mathcal{G} = (\mathcal{C}, N, S, \mathcal{R})$ where:

- 1 $\mathcal{C} = (\Sigma, \text{type})$ is a signature;
- 2 $N \subseteq \Sigma$ is the set of non-terminal labels;
- 3 $S \in N$ is the start symbol;
- 4 \mathcal{R} is a finite set of rules over N ;

with $\max(\{\text{type}(R) \mid (L, R) \in \mathcal{R}\}) \leq k$. The generated language is:

$$L(\mathcal{G}) = \{H \in \mathcal{H}_{\mathcal{C}} \mid S^{\bullet} \Rightarrow_{\mathcal{R}}^* H \text{ with } \text{lab}_H^{-1}(N) = \emptyset\} \subseteq \mathcal{H}_{\mathcal{C}}.$$

$L \subseteq \mathcal{H}_{\mathcal{C}}$ is called a HR language of order k (k -HR language) if there is a k -HR grammar such that $L(\mathcal{G}) = L$.

The class of HR languages is the union of all k -HR languages for $k \in \mathbb{N}$.

HRS Languages

Definition (String Graph Language)

A hypergraph language that only consists of string graphs is called a string graph language. We can identify the language members with the strings they represent.

Thus, a string language L is called a HRS language if, up to treatment of the empty string, there is a HR grammar that generates the language of string graphs that represent exactly the strings in L .

The class of HRS languages is the union of all k -HRS languages for $k \in \mathbb{N}$.

Theorem (Precise Theorem 5)

For all $k \geq 1$, $\mathcal{HRS}_{2k} = \mathcal{HRS}_{2k+1} = \mathcal{MCF}_k$.

Parallel Derivations

Definition (Parallel Direct Derivation)

Let $H \in \mathcal{H}_C$ with $E_H = \{e_1, \dots, e_n\}$, and \mathcal{R} be a set of rules. If for each $e_i \in E_H$, there is an $R_i \in \mathcal{H}_C$ such that $(\text{lab}_H(e_i), R_i) \in \mathcal{R}$, then H parallelly directly derives $H' \cong H[e_1/R_1, \dots, e_n/R_n]$, and write $H \Rightarrow_{\mathcal{R}} H'$.

Definition (Parallel Derivation)

Let $\mathcal{S} = \{\mathcal{R}_i \mid i \in I\}$ be a finite set of rule sets indexed by I , and \mathcal{M} an FSA over I . Then H (\mathcal{M} -)parallelly derives H' if there is a sequence $H \Rightarrow_{\mathcal{R}_{i_1}} H_1 \Rightarrow_{\mathcal{R}_{i_2}} \dots \Rightarrow_{\mathcal{R}_{i_k}} H_k \cong H'$ ($k \in \mathbb{N}$) such that $i_1 i_2 \dots i_k \in L(\mathcal{M})$. We write $H \Rightarrow_{\mathcal{S}}^{\mathcal{M}} H'$, $H \Rightarrow_{\mathcal{S}}^{i_1 i_2 \dots i_k} H'$ or $H \Rightarrow_{\mathcal{S}}^k H'$.

Definition (Table)

A table T is a finite set of rules over Σ such that for each $L \in \Sigma$, there is at least one $R \in \mathcal{H}_C$ such that $(L, R) \in T$.

PHR Grammars I

Definition (PHR Grammar)

A (k -PHR grammar is a system $\mathcal{G} = (\mathcal{C}, A, S, \mathcal{T}, \mathcal{M})$ where:

- 1 $\mathcal{C} = (\Sigma, \text{type})$ is a signature;
- 2 $A \subseteq \Sigma$ is the set of terminal labels;
- 3 $S \in \Sigma \setminus A$ is the start symbol;
- 4 $\mathcal{T} = \{T_i \mid i \in I\}$ is a finite set of tables indexed by I ;
- 5 $\mathcal{M} = (Q, I, \delta, i, F)$ is an FSA over I ;

with $\max(\{|type|(R) \mid (L, R) \in \bigcup_{T_i \in \mathcal{T}} T_i\}) \leq k$. The generated language is:

$$L(\mathcal{G}) = \{H \in \mathcal{H}_C \mid S^\bullet \Rightarrow_{\mathcal{T}}^{\mathcal{M}} H \text{ with } \text{lab}_H^{-1}(A) = E_H\} \subseteq \mathcal{H}_C.$$

The inclusion of rational control, originally considered for ET0L by Asveld 1977, is a convenience to aid with the specification of human understandable grammars. This actually adds no power...

PHR Grammars II

Lemma (Control Removal)

Given a k -PHR grammar \mathcal{G} , one can effectively construct a k -PHR grammar \mathcal{G}' without control such that $L(\mathcal{G}) = L(\mathcal{G}')$.

Proof: Encode control in the labels! Make a copy of all the labels for all of the states in the FSA, and moving between control states is synchronized with moving between the copies of labels. □

Proposition

$L = \{a^{2^n} \mid n \in \mathbb{N}\}$ is an ETOL language but not MCF or semilinear.

Theorem (PHR Generalises HR)

For $k \geq 0$, $\mathcal{HR}_k \subsetneq \mathcal{PHR}_k$.

Proof: Induction on derivation length. □

PHRS Languages

Lemma (PHRS Generalises ETOL)

$ETOL = PHRS_2$ and for $k \geq 4$, $ETOL \subsetneq PHRS_k$.

Proof: Forward direction follows from by transforming 2-PHR such that all labels have at least type 2 and all RHS hypergraphs have proper edges. Reverse follows from transforming ETOL grammars to EPTOL. \square

Corollary

There are 2-PHRS languages that are not semilinear.

Lemma (PHRS Generalises MCF)

For $k \geq 2$, $HRS_k \subsetneq PHRS_k$.

Proof: The equivalence of MCF and HRS due to Theorem 15. To see the remainder follows from Theorem 22 and its proof. We get strictness from Proposition 21 together with Lemma 23. \square

Closing Conjectures

Conjecture (CS Generalises PHRS)

$\mathcal{PHRS} \subsetneq \mathcal{CS}$.

Conjecture (Substitution-Closed Full AFL)

For $k \geq 2$, \mathcal{PHRS}_k and \mathcal{PHRS} are substitution-closed full AFLs.

Conjecture (WP Double Torus)

The fundamental group of the double torus admits a PHRS word problem with is neither a MCF nor ET0L language.

Corollary

The word problem of any surface group is a PHRS language.

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