Confluence up to Garbage

13th International Conference on Graph Transformation

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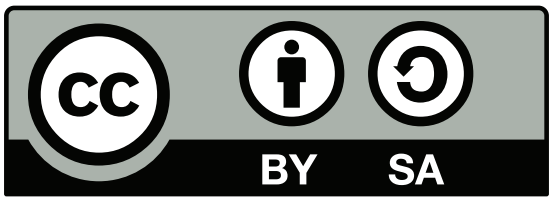

Motivating Question

When does a graph grammar admit a deterministic algorithm for deciding membership?

Graph Languages

Setting: DPO graph transformation with:

- all rules and matches are injective;
- systems with finitely many rules.

Definition (Graph Grammar)

Given a GT system $T = (\Sigma, \mathcal{R})$, a subalphabet of *non-terminals* $\mathcal{N}$, and a *start graph* S over Σ, then a grammar is a tuple $(\Sigma, \mathcal{N}, \mathcal{R}, S)$.

$$\mathrm{L}(\mathcal{G}) = \{G \mid S \Rightarrow^*_{\mathcal{R}} G, G \text{ terminally labelled}\}.$$

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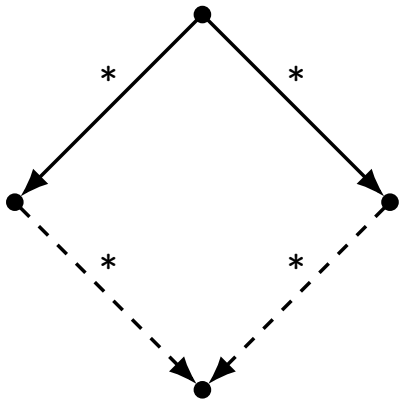
- $G \Rightarrow_r H$ if and only if $H \Rightarrow_{r^{-1}} G$, for some $r \in \mathcal{R}$.
- $G \in \mathrm{L}(\mathcal{G})$ if and only if $G \Rightarrow^*_{\mathcal{R}^{-1}} S$ and G is terminally labelled.
- If $\mathcal{R}^{-1}$ terminates, then we have a non-deterministic membership checking algorithm.

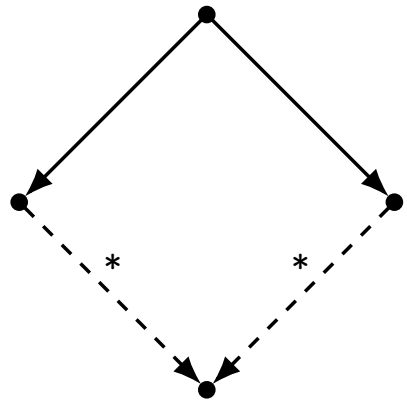
Example: Recognising Trees

Starting from a single node, the following DPO rule can be used to generate the language of trees:

The reversed system is confluent and terminating, so no backtracking is required to check if a graph is a tree.



(a) Confluence



(b) Local Confluence

Results about Confluence

Theorem (Plump (1998) and Plump (2005))

It is undecidable, in general:

- whether a GT system is terminating;
- whether a terminating GT system is confluent.

Lemma (Plump (2005))

A GT system is locally confluent if (but not only if) all its critical pairs are strongly joinable.

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Lemma (Newman (1942))

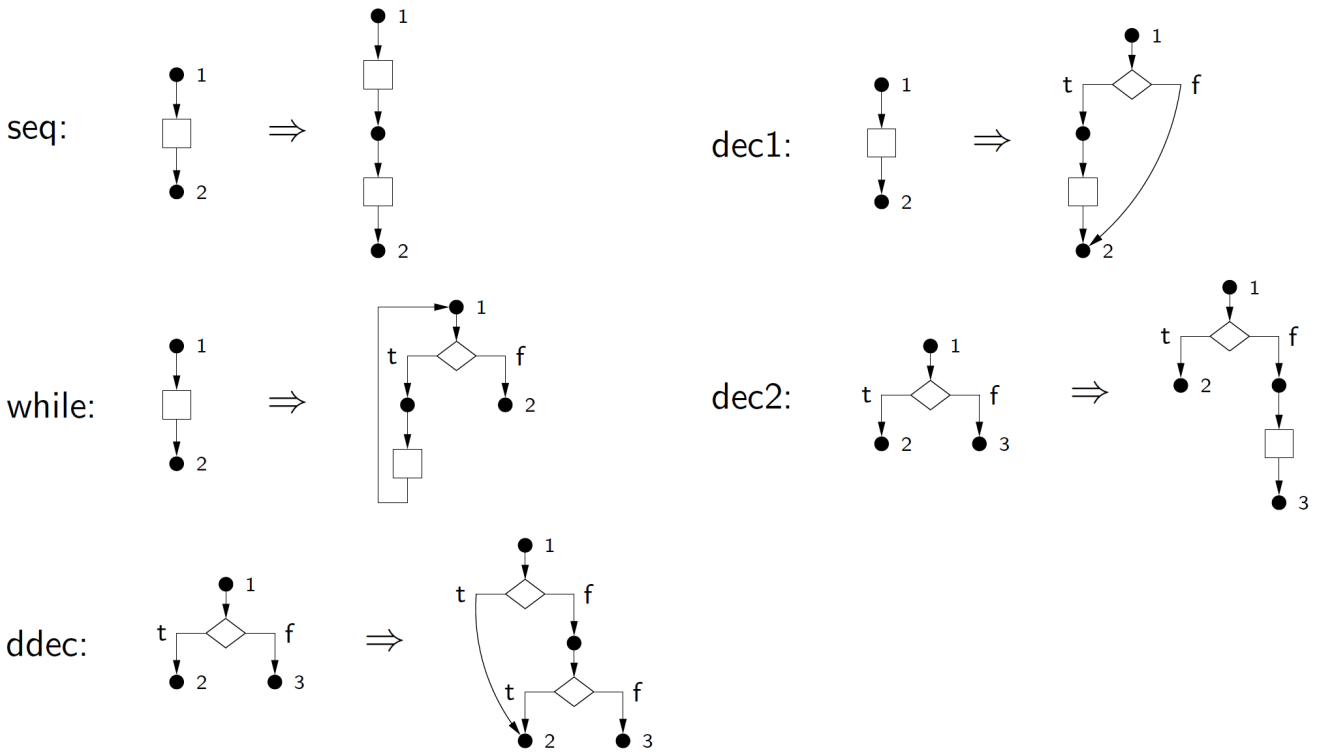
In the presence of termination, confluence and local confluence coincide.

Given termination, we may be able to automatically detect confluence.

Example: Extended Flow Diagrams I

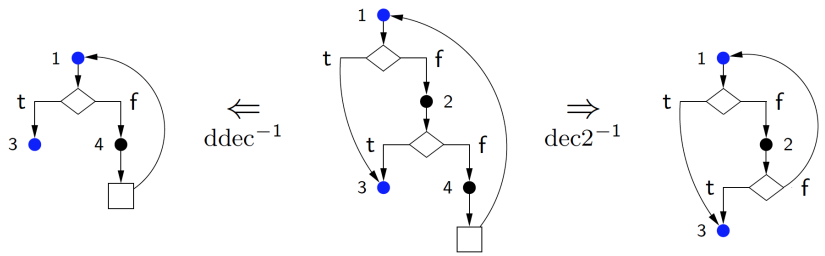
The language of EFDs is a is contained in the language semi-structured flow graphs (Farrow, Kennedy, and Zucconi 1976; Plump 2005).

The language is generated by:



Example: Extended Flow Diagrams II

The inverse rules are not confluent. Non-joinable pair:



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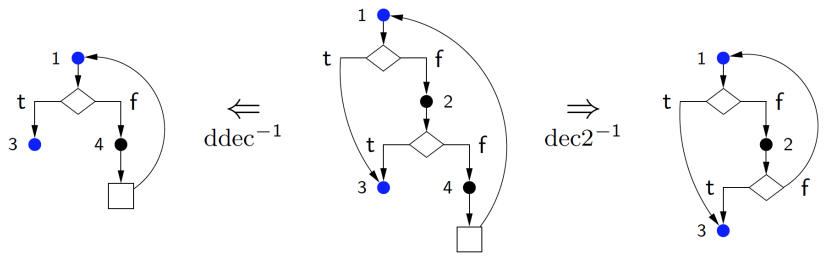
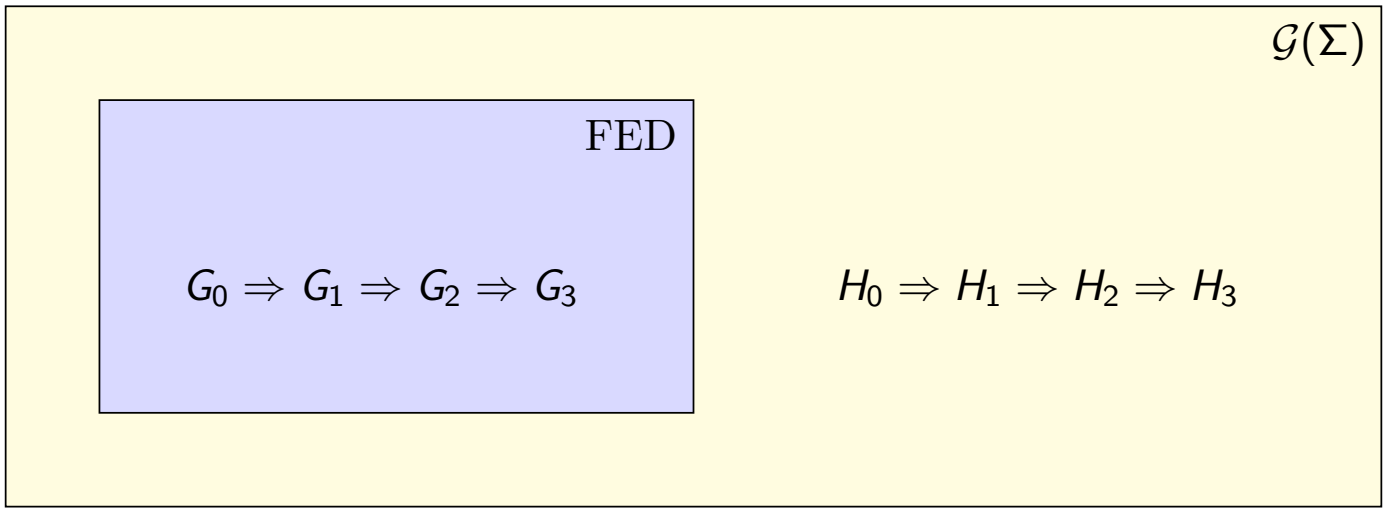
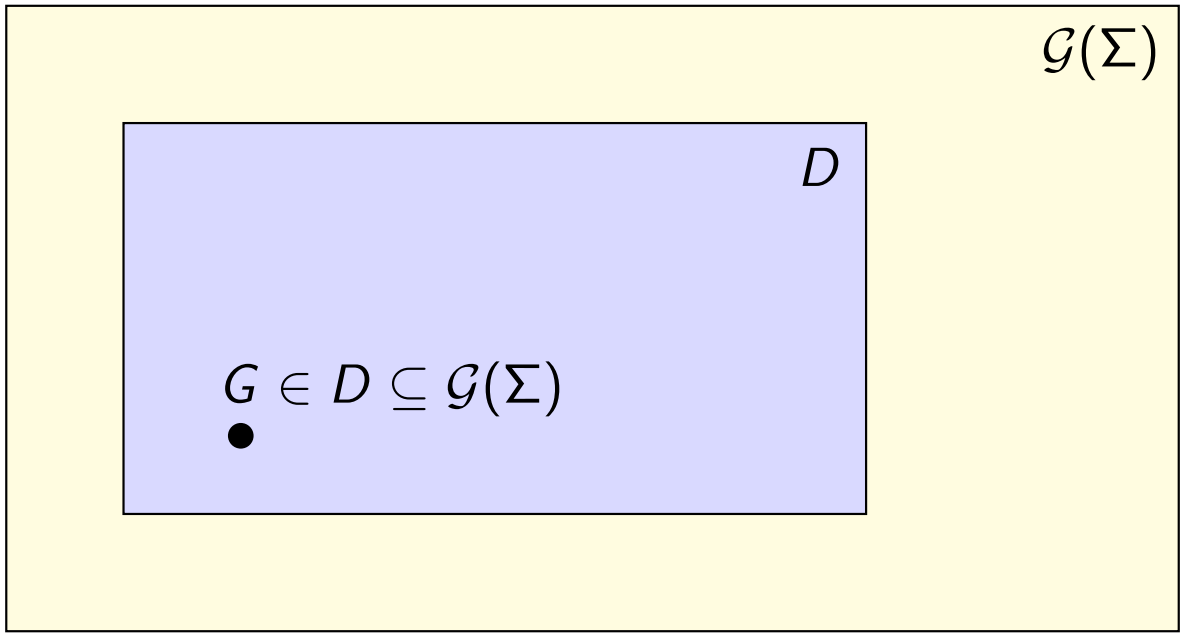


It will turn out that we still have deterministic membership checking:



Confluence up to Garbage

Confluence up to garbage only considers start graphs in some language D.



Theorem

Given a grammar $\mathcal{G}$ with rules $\mathcal{R}$, termination and confluence up to garbage of $\mathcal{R}^{-1}$ give us deterministic membership checking for $\mathrm{L}(\mathcal{G})$.

Non-Garbage Critical Pairs

Definition (Critical Pair)

A critical pair of $(\Sigma, \mathcal{R})$ is a pair of parallelly dependent direct derivations $H_1 \Leftarrow_{r_1,g_1} G \Rightarrow_{r_2,g_2} H_2$ such that $G = g_1(L_1) \cup g_2(L_2)$ and if $r_1 = r_2$ then $g_1 \neq g_2$.

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Definition (Non-Garbage Critical Pair)

Let $T = (\Sigma, \mathcal{R})$ and $\mathcal{D} \subseteq \mathcal{G}(\Sigma)$. A $\mathcal{D}$-non-garbage critical pair of T is a critical pair $H_1 \Leftarrow_{r_1,g_1} G \Rightarrow_{r_2,g_2} H_2$ such that G is in $\widehat{\mathcal{D}}$, the subgraph closure of $\mathcal{D}$.

If $\mathcal{D}$ has decidable subgraph closure, then we can effectively enumerate the (finitely many) non-garbage critical pairs.

Generalised Critical Pair Lemma I

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A critical pair is strongly joinable if it is joinable without deleting any of the persistent nodes, and the persistent nodes are identified when joining.

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Theorem (Generalised Critical Pair Lemma)

Let $\mathcal{T} = (\Sigma, \mathcal{R})$ and $\mathcal{D} \subseteq \mathcal{G}(\Sigma)$. If all $\mathcal{T}$'s $\mathcal{D}$-non-garbage critical pairs are strongly joinable, then $\mathcal{T}$ is locally confluent on $\mathcal{D}$.

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Theorem (Generalised Critical Pair Lemma)

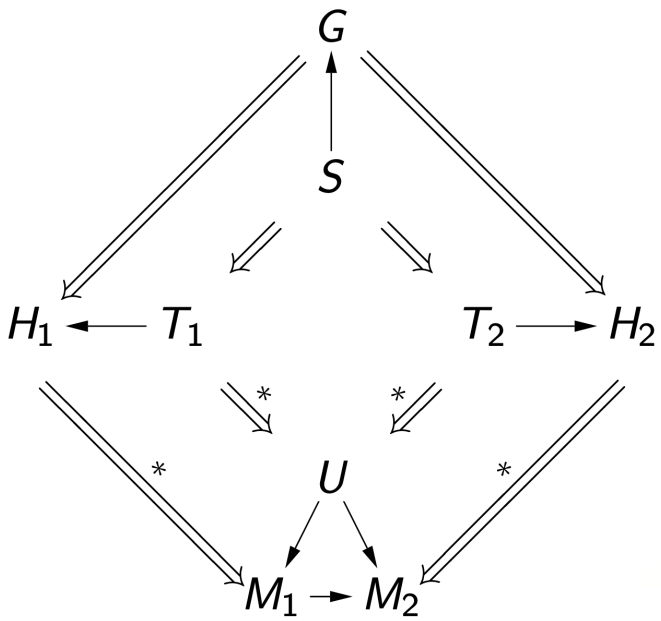
Let $T = (\Sigma, \mathcal{R})$ and $\mathcal{D} \subseteq \mathcal{G}(\Sigma)$. If all T's $\mathcal{D}$-non-garbage critical pairs are strongly joinable, then T is locally confluent on $\mathcal{D}$.

Proof sketch: We need to show that every pair of derivations $H_1 \Leftarrow_{r_1,g_1} G \Rightarrow_{r_2,g_2} H_2$ such that G is non-garbage can be joined.

Case 1: If parallelly independent, then done by commutativity (Ehrig and Kreowski 1976).

Generalised Critical Pair Lemma II

Case 2: The Clipping Theorem (Kreowski 1977) tells us how to factor out a minimal derivation $T_1 \Leftarrow S \Rightarrow T_2$. This must be a critical pair, and $G \in \mathcal{D}$, then $S \in \widehat{\mathcal{D}}$, so the critical pair is non-garbage. Non-garbage pairs are strongly joinable (by assumption), so we can use the Embedding Theorem (Ehrig 1977) to get our result.

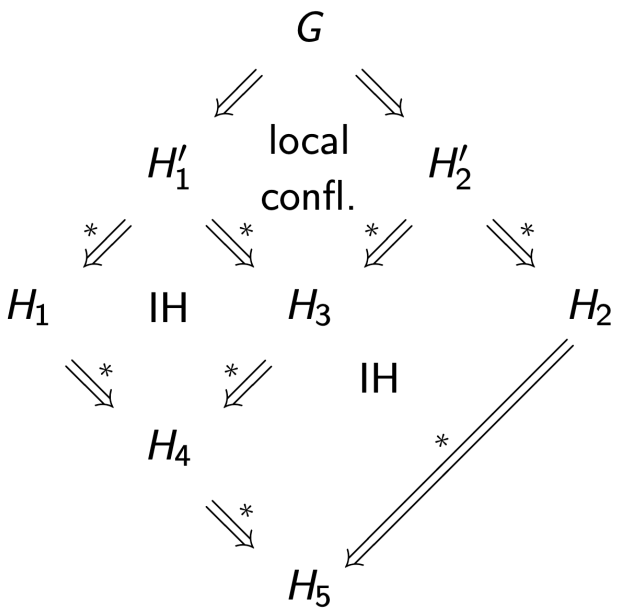


Generalised Newman's Lemma

Lemma (Generalised Newman's Lemma)

Let $T = (\Sigma, \mathcal{R})$ and $\mathcal{D} \subseteq \mathcal{G}(\Sigma)$. If T is terminating on $\mathcal{D}$ and $\mathcal{D}$ is closed under T, then T is confluent on $\mathcal{D}$ if and only if is locally confluent on $\mathcal{D}$.

Proof sketch: Noetherian induction.



Checking for Confluence up to Garbage

Input:

1. GT system T.
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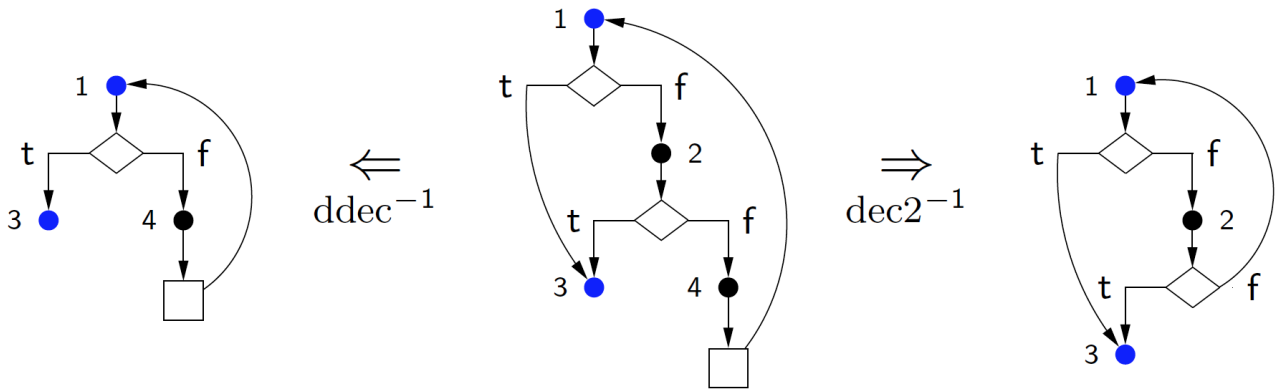
Process:

1. Establish that T is terminating on $\mathcal{D}$ and $\mathcal{D}$ is closed under T. If closedness cannot be established, restart with a superset of D.
2. Generate the finitely many non-garbage critical pairs of T.
3. Check if each generated pair is strongly joinable:
   - All pairs are strongly joinable: confluence on $\mathcal{D}$;
   - One of the pairs is not joinable: a direct counter example;
   - All pairs are joinable, but not all strongly: inconclusive.

Example: Extended Flow Diagrams III

Recall we had a non-joinable pair:



Example: Extended Flow Diagrams III

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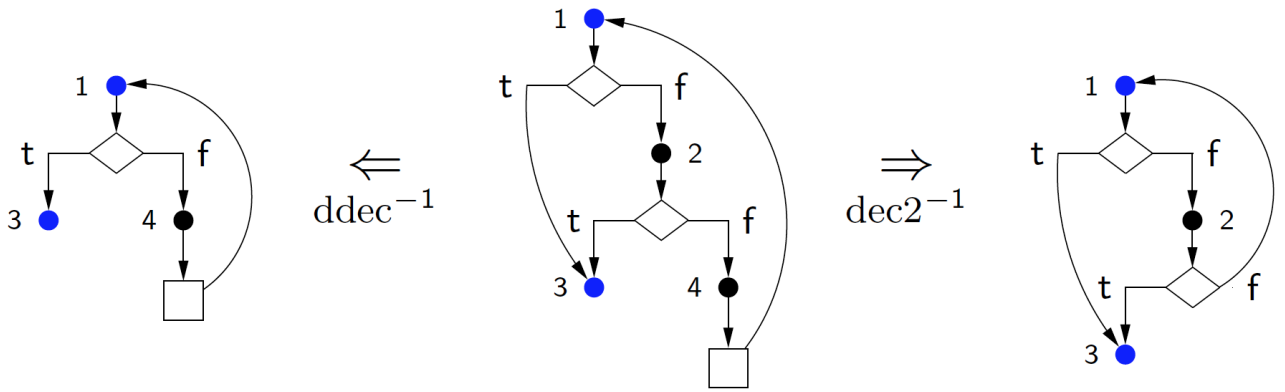

- The start graph of the pair cannot be embedded in an EFD.
- The pair is garbage, and all others are strongly joinable.
- We have termination and confluence up to garbage.
- Thus, we have determinsitic recognition.

Example: Series Parallel Graphs I

Series-parallel graphs were introduced by Duffin (1965). It is the smallest class of graphs closed under *parallel* and *sequential* composition, containing $P = \overset{s}{\bullet}\!\!\rightarrow\!\!\overset{t}{\bullet}$, where s is the *source* and t the *sink*.

Parallel composition identifies the two *sources* and *sinks*, and *sequential* composition identifies the *sink* of one with the *source* of another.

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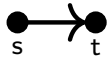
Series-parallel graphs were introduced by Duffin (1965). It is the smallest class of graphs closed under *parallel* and *sequential* composition, containing $P =$ 

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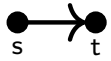
Series-parallel graphs were introduced by Duffin (1965). It is the smallest class of graphs closed under *parallel* and *sequential* composition, containing $P =$ 

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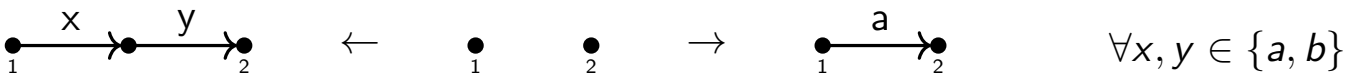
Duffin showed that a graph is series-parallel if and only if it can be generated from P by the DPO rules:

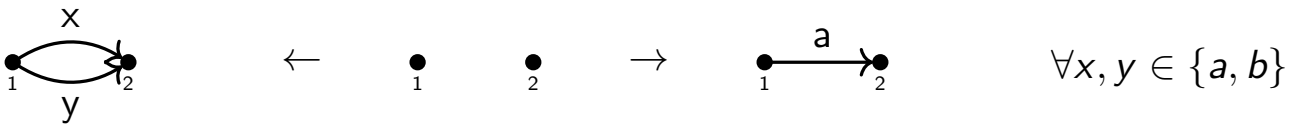
The reversed rules are confluent and terminating, however introducing labels to the edges breaks confluence (Hristakiev and Plump 2018), but we can still establish confluence up to garbage...

Example: Series Parallel Graphs II

Reduction rules for the language of labelled series-parallel graphs:

s_i: 

$\forall x, y \in \{a, b\}$

p_i: 

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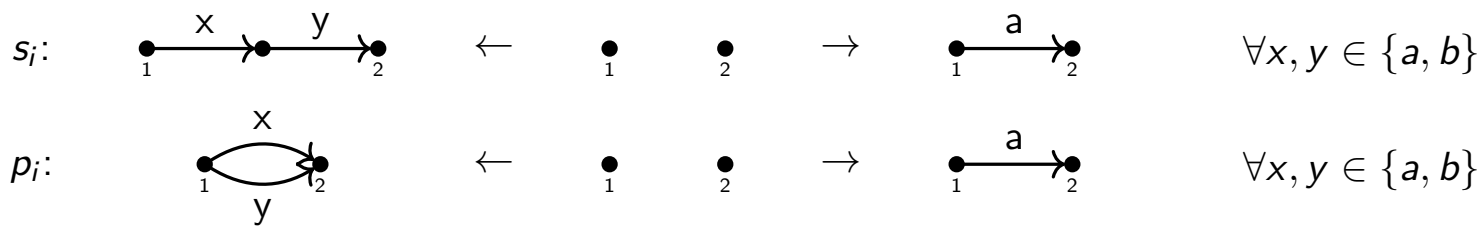
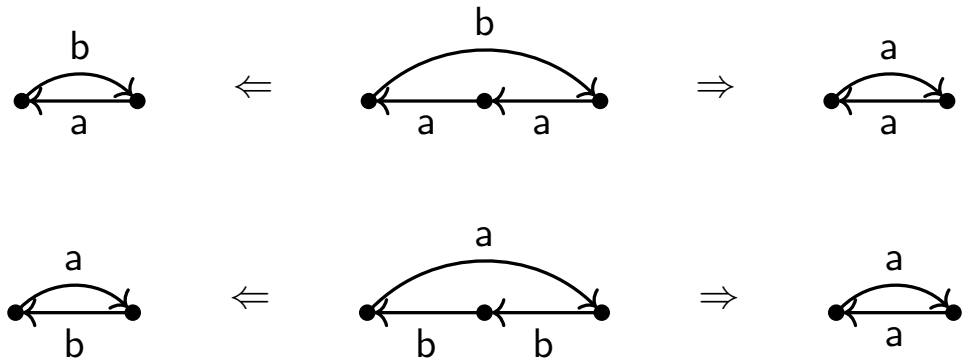


All critical pairs are strongly joinable other than the garbage ones:



We have termination and confluence up to garbage, so determinsitic recognition, as before.

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- The Generalised Newman's Lemma works for any ARS.
- The Generalised Critical Pair Lemma works:
  - for (labelled) hypergraphs too, and we can relax injectivity of rules such that $K \to R$ need not be injective;
  - for any adhesive system with the usual restrictions, though the proof is a little different (Campbell 2019);
  - in the setting of (hyper)graph transformation with node relabelling (with root nodes) (not ($\mathcal{M}$-)adhesive!) (Campbell and Plump 2019).

Related and Future Work

- Usefulness of classification of non-garbage conflicts (Lambers, Born, Orejas, Strüber, and Taentzer 2018; Lambers, Kosiol, Strüber, and Taentzer 2019)?

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- Connection to graphs satisfying negative constraints (Lambers 2009)?
- Find checkable sufficient conditions which allow to decide membership in the subgraph closure of a language.
- Connection to confluence modulo garbage (work in progress).

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