Efficient Graph Rewriting

York Semigroup

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May 2019

Unlabelled Graphs I

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We can formally define a **concrete graph** as:

$$G = (V, E, s : E \to V, t : E \to V)$$

where V is a **finite** set of **vertices**, E is a **finite** set of **edges**. We call $s : E \to V$ the **source** function, and $t : E \to V$ the **target** function.

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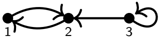
Example: $G = (\{1, 2, 3\}, \{a, b, c, d\}, s, t)$ where $s = \{(a, 1), (b, 2), (c, 3), (d, 3)\}$, $t = \{(a, 2), (b, 1), (c, 1), (d, 3)\}$.



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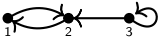

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A **graph morphism** $g : G \to H$ is a pair $(g_V : V_G \to V_H, g_E : E_G \to E_H)$ such that sources and targets are preserved. That is, $\forall e \in E_G$, $g_V(s_G(e)) = s_H(g_E(e))$ and $g_V(t_G(e)) = t_H(g_E(e))$.

Unlabelled Graphs II

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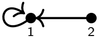
$G =$  $H =$ 

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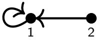
$G =$ 

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There are four morphisms $G \to H$, three of which are injective, none of which are surjective. There are actually also four morphisms $H \to G$, three of which are surjective.

Definition 4

We say that graphs G, H are **isomorphic** iff there exists a **graph isomorphism** $g : G \to H$, and we write $G \cong H$. This naturally gives rise to **equivalence classes** $[G]$, called **abstract graphs**.

Labelled Graphs I

Definition 5

A label alphabet $\mathcal{L} = (\mathcal{L}_V, \mathcal{L}_E)$ consists of **finite** sets of **node labels** $\mathcal{L}_V$ and **edge labels** $\mathcal{L}_E$.

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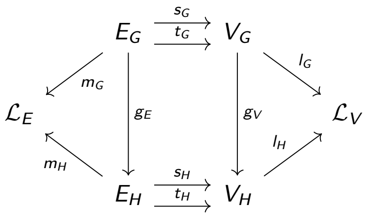
Definition 6

A **concrete labelled graph** over a label alphabet $\mathcal{L}$ is a tuple $G = (V, E, s, t, l, m, p)$ where:

1. V is a **finite** set of **vertices**;
2. E is a **finite** set of **edges**;
3. $s : E \to V$ is a source function;
4. $t : E \to V$ is a target function;
5. $l : V \to \mathcal{L}_V$ is the node labelling function;
6. $m : E \to \mathcal{L}_E$ is the edge labelling function;

Labelled Graphs II

For morphisms between labelled graphs, we require that labels are preserved: $\forall v \in V_G,\ l_G(v) = l_H(g_V(v))$ and $\forall e \in E_G,\ m_G(e) = m_H(g_E(e))$.



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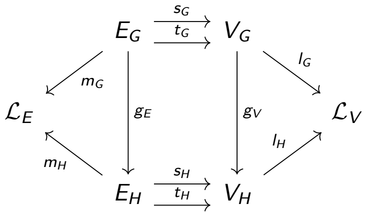

Definition 7

Given a common alphabet $\mathcal{L}$, we say H is a **subgraph** of G iff there exists an **inclusion morphism** $H \hookrightarrow G$. This happens iff $V_H \subseteq V_G$, $E_H \subseteq E_G$, $s_H = s_G|_{E_H}$, $t_H = t_G|_{E_H}$, $l_H = l_G|_{V_H}$, $m_H = m_G|_{E_H}$.

Rules

Let $\mathcal{L} = (\mathcal{L}_V, \mathcal{L}_E)$ be the ambient label alphabet, and graphs be concrete.

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A **rule** $r = \langle L \leftarrow K \rightarrow R \rangle$ consists of **labelled graphs** L, K, R over $\mathcal{L}$, and **inclusions** $K \hookrightarrow L$ and $K \hookrightarrow R$.

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If $r = \langle L \leftarrow K \rightarrow R \rangle$ is a **rule**, then $|r| = max\{|L|, |R|\}$, where the size of a graph G is $|G| = |V_G| + |E_G|$.

Rule Application

Definition 11

Given a **rule** $r = \langle L \leftarrow K \rightarrow R \rangle$ and a **labelled graph** G, we say that an **injective** morphism $g : L \hookrightarrow G$ satisfies the **dangling condition** iff no edge in $G \setminus g(L)$ is incident to a node in $g(L \setminus K)$.

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Definition 12

To **apply** a rule r to some **labelled graph** G, find an **injective** graph morphism $g : L \hookrightarrow G$ satisfying the **dangling condition**, then:

1. Delete $g(L \setminus K)$, giving the **intermediate graph** D;
2. Add disjointly $R \setminus K$ to D, giving the **result graph** H.

If the **dangling condition** fails, the rule is not applicable using **match** g. We can exhaustively check all matches to determine applicability.

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We write $G \Rightarrow_{r,g} M$ for a successful application of r to G using match g, obtaining result $M \cong H$. We call this a **direct derivation**.

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It turns out that **deletions** *are* **natural pushout complements** *and* **gluings** *are* **natural pushouts** *in the category of labelled graphs. Moreover, direct derivations are* **natural double pushouts**, *D and H are* **unique up to isomorphism**, *and derivations $G \Rightarrow_{r,g} H$ are* **invertible**.

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Definition 15

For a given set of rules $\mathcal{R}$, we write $G \Rightarrow_{\mathcal{R}} H$ iff H is **directly derived** from G using any of the rules from $\mathcal{R}$.

Graph Transformation

Definition 16

A **graph transformation system** (**GT system**) is a pair $T = (\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is a **label alphabet** and $\mathcal{R}$ is a **finite** set of **rules**.

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Let $T = (\mathcal{L}, \mathcal{R})$ be a **GT system**. Then $(\mathcal{G}(\mathcal{L}), \to_{\mathcal{R}})$ is the induced **ARS** defined by $\forall [G], [H] \in \mathcal{G}(\mathcal{L}), [G] \to_{\mathcal{R}} [H]$ iff $G \Rightarrow_{\mathcal{R}} H$.

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Lemma 19

Consider the **ARS** $(\mathcal{G}(\mathcal{L}), \to)$ *induced by a* **GTS**. *Then* $\to$ *is a* **binary relation** *on* $\mathcal{G}(\mathcal{L})$ *(that is, it is both well-defined and closed). Moreover, it is* **finitely branching** *and* **decidable**.

Graph Grammars

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Given a **GT system** $T = (\mathcal{L}, \mathcal{R})$, a subalphabet of **non-terminals** $\mathcal{N}$, and a **start graph** S over $\mathcal{L}$, then a **graph grammar** is the system $\boldsymbol{G} = (\mathcal{L}, \mathcal{N}, \mathcal{R}, S)$.

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Given a **graph grammar** $\boldsymbol{G}$ as defined above, we say that a graph G is **terminally labelled** iff $l(V) \cap \mathcal{N}_V = \emptyset$ and $m(E) \cap \mathcal{N}_E = \emptyset$. Thus, we can define the **graph language** generated by $\boldsymbol{G}$:

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Theorem 22 (Membership Test)

Given a **grammar** $\boldsymbol{G} = (\mathcal{L}, \mathcal{N}, \mathcal{R}, S)$, $[G] \in \boldsymbol{L}(\boldsymbol{G})$ *iff* $[G] \rightarrow^*_{\mathcal{R}^{-1}} [S]$ *and* G *is terminally labelled.*

TREE Language

Let ***TREE*** $= (\mathcal{L}, \mathcal{N}, S, \mathcal{R})$ where:

1. $\mathcal{L} = (\{\square\}, \{\square\})$ where $\square$ denotes the empty label;
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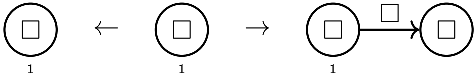

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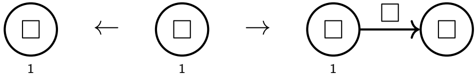

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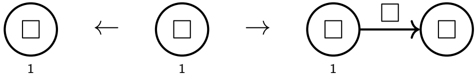

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One can see (via critical pair analysis) that $\textit{TREE}^{-1} = (\mathcal{L}, \{r^{-1}\})$ is confluent too... QUEUE WAFFLE

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Lemma 25

The GMP requires $O(|G|^{|L|})$ time. The RAP requires $O(|r|)$ time.

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BUT... what goes wrong...

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Our graph morphisms would then need to more strongly preserve rootedness of nodes. That is, and unrooted node can no longer be mapped to a rooted node. Only a node of undefined rootedness can be changed... Rather similar to the trick with partial labelling.

Rooted Graphs

Definition 26

A **graph** over $\mathcal{L}$ is a tuple $G = (V, E, s, t, l, m, p)$ where:

1. V is a **finite** set of **vertices**;
2. E is a **finite** set of **edges**;
3. $s : E \to V$ is a **total** source function;
4. $t : E \to V$ is a **total** target function;
5. $l : V \to \mathcal{L}_V$ is a **partial** function, labelling the vertices;
6. $m : E \to \mathcal{L}_E$ is a **total** function, labelling the edges;
7. $p : V \to \mathbb{Z}_2$ is a **partial** function, determining vertex rootedness.

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Definition 27

A graph G is **totally labelled** iff l_G is total, and **totally rooted** if p_G is total. If G is both, then we call it a **TLRG**.

Rooted Morphisms

Definition 28

A **graph morphism** between graphs G and H is a pair of functions $g = (g_V : V_G \to V_H, g_E : E_G \to E_H)$ such that sources, targets, labels, and rootedness are preserved. That is:

1. $\forall e \in E_G,\ g_V(s_G(e)) = s_H(g_E(e))$;
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All of the other theory we've seen for the standard case also holds... transformation occurs on the TLRGs with K partially labelled and partially rooted.

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Given a **TLRG** *G of* **bounded degree** *containing a* **bounded** *number of root nodes, and a GT system $T = (\mathcal{L}, \mathcal{R})$ where each rule is* **fast**, *then one can decide in* **constant time** *the* **direct successors** *of $[G]$.*

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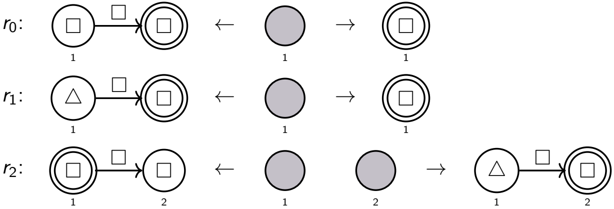
Corollary 31

If each rule is additionally **root non-increasing** *and* **degree non-increasing**, *and* T **terminating** *with maximum derivation length* $N \in \mathbb{N}$, *then one can find a* **normal form** *of* $[G]$ *in* $O(N)$ *time.*

Recognising Trees I

Let $\mathcal{L} = (\{\square, \triangle\}, \{\square\})$, and $\mathcal{R} = \{r_0, r_1, r_2\}$.



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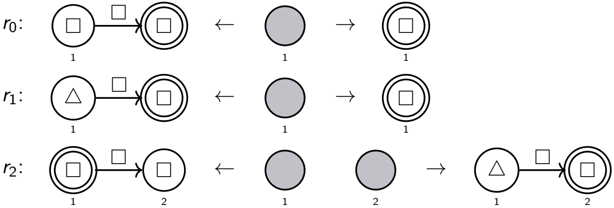

Intuitively, this works by pushing the “root” to the bottom of a branch, and then pruning. If we start with a tree and run this until we cannot do it anymore, we must be left with a single node.

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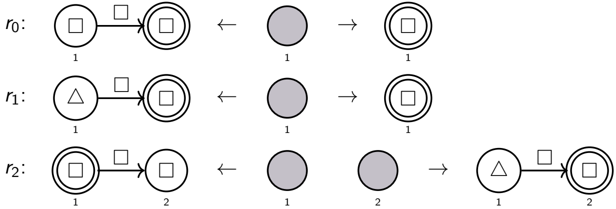

Intuitively, this works by pushing the “root” to the bottom of a branch, and then pruning. If we start with a tree and run this until we cannot do it anymore, we must be left with a single node.

The triangle labels are necessary so that, in the case that the input graph is not a tree, we could “get stuck” in a directed cycle.

Recognising Trees II

Definition 32

Given a graph G, we define $G^{\ominus}$ to be exactly G, but with every node unrooted, and everything labelled by $\square$. That is, $G^{\ominus} = (V_G, E_G, s_G, t_G, V_G \times \{\square\}, E_G \times \{\square\}, V_G \times \{0\})$.

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Definition 33

By "input graph", we mean any TLRG containing exactly one "root" node, with edges and vertices all labelled by $\square$.

Theorem 34 (Tree Recognition)

Given an input graph G, one may use the system $(\mathcal{L}, \mathcal{R})$ from G to find a normal form for G, say H. H is the single root-node graph labelled by $\square$ iff $[G^{\ominus}] \in \boldsymbol{L}(TREE)$. Moreover, for input graphs of bounded degree, we terminate in linear time (with respect to $|V_G|$).

Motivation

Notice in the rooted tree example that every pair $H_1 \Leftarrow G \Rightarrow H_2$ where G was a tree can be joined, but this is not necessarily true of any G in general, so we don't have local confluence.

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Definition 35

Let $T = (\mathcal{L}, \mathcal{R})$ be a GT system, and $D \subseteq \mathcal{G}(\mathcal{L})$ be a set of abstract graphs. Then, a graph G is called **garbage** iff $[G] \notin D$.

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Definition 36

Let $T = (\mathcal{L}, \mathcal{R})$, and $D \subseteq \mathcal{G}(\mathcal{L})$. T is **weakly garbage separating** w.r.t. D iff for all G, H such that $G \Rightarrow_{\mathcal{R}} H$, if $[G] \in D$ then $[H] \in D$. T is **garbage separating** iff we have $[G] \in D$ iff $[H] \in D$.

Confluence Modulo Garbage

Definition 37

Let $T = (\mathcal{L}, \mathcal{R})$, $D \subseteq \mathcal{G}(\mathcal{L})$. If for all graphs G, H_1, H_2, such that $[G] \in D$, if $H_1 \Leftarrow_{\mathcal{R}}^* G \Rightarrow_{\mathcal{R}}^* H_2$ ($H_1 \Leftarrow_{\mathcal{R}} G \Rightarrow_{\mathcal{R}} H_2$) implies that H_1, H_2 are **joinable**, then T is (**locally**) **confluence modulo garbage** w.r.t. D.

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Definition 38

Let $T = (\mathcal{L}, \mathcal{R})$, $D \subseteq \mathcal{G}(\mathcal{L})$. If there is no infinite derivation sequence $G_0 \Rightarrow_{\mathcal{R}} G_1 \Rightarrow_{\mathcal{R}} G_2 \Rightarrow_{\mathcal{R}} \cdots$ such that $[G_0] \in D$, then T is **terminating modulo garbage** w.r.t. D.

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Theorem 39 (Newman-Garbage Lemma)

Let $T = (\mathcal{L}, \mathcal{R})$, $D \subseteq \mathcal{G}(\mathcal{L})$. If T is **terminating modulo garbage** *and* **weakly garbage separating**, *then it is* **confluent modulo garbage** *iff it is* **locally confluent modulo garbage**.

Tree Recognition Revisited

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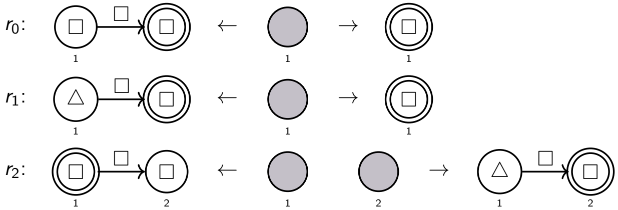

Lemma 40

The GT system $T = (\mathcal{L}, \mathcal{R})$ *is* **garbage separating** *w.r.t. to* $D = \{[G] \in \mathcal{G}^{\oplus}(\mathcal{L}) \mid [G^{\ominus}] \in \mathbf{L}(\textbf{\textit{TREE}}), |p_G^{-1}(\{1\})| = 1\}$ *and* **confluent modulo garbage** *w.r.t.* $E = \{[G] \in D \mid l_G(V_G) = \{\square\}\}$.

Showing Confluence

It is well known that for totally labelled systems (that is, the interface graph K is totally labelled), that it is sufficient (but not necessary) to check “strong joinability” of “critical pairs”.

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Theorem 41 (Non-Garbage Critical Pair Lemma)

Let $T = (\mathcal{L}, \mathcal{R})$, $D \subseteq \mathcal{G}(\mathcal{L})$. *If all its* **non-garbage critical pairs** *are* **strongly joinable**, *then* T *is* **locally confluent mod garbage** *w.r.t.* D.

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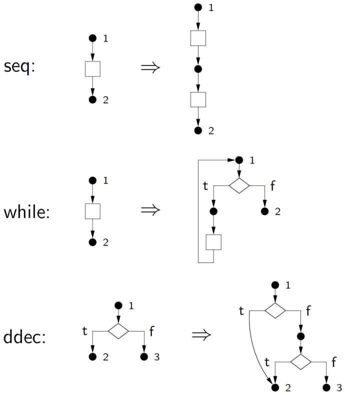
Corollary 42

Let $T = (\mathcal{L}, \mathcal{R})$, $D \subseteq \mathcal{G}(\mathcal{L})$. *If* T *is* **terminating modulo garbage**, **weakly garbage separating**, *and all its* **non-garbage critical pairs** *are* **strongly joinable** *then* T *is* **confluent modulo garbage**.

Extended Flow Diagrams I

The language of **extended flow diagrams** is generated by $\textbf{\textit{EFD}} = (\mathcal{L}, \mathcal{N}, \mathcal{R}, S)$ where $\mathcal{L}_V = \{\bullet, \square, \diamond\}$, $\mathcal{L}_E = \{t, f, \square\}$, $\mathcal{N}_V = \mathcal{N}_E = \emptyset$, $\mathcal{R} = \{seq, while, ddec, dec1, dec2\}$, and $S = \bullet\!\!\rightarrow\!\square\!\!\rightarrow\!\!\bullet$.



Extended Flow Diagrams II

Lemma 43

$EFD^{-1} = (\mathcal{L}, \mathcal{R}^{-1})$ is **terminating**. *Moreover, it is* **garbage separating** *w.r.t.* $\boldsymbol{L}(\boldsymbol{EFD})$.

Lemma 44

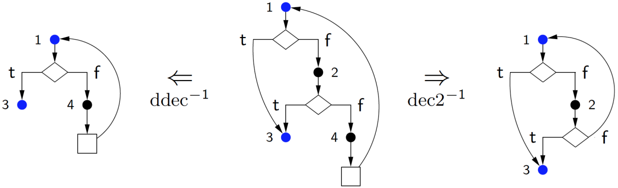
Every **directed cycle** *in a graph in the* **subgraph closure** *of* $\boldsymbol{L}(\boldsymbol{EFD})$ *contains a t-labelled edge.*

Theorem 45 (EFD Recognition)

$EFD^{-1} = (\mathcal{L}, \mathcal{R}^{-1})$ is **confluent modulo garbage** *w.r.t.* $\boldsymbol{L}(\boldsymbol{EFD})$, *but not* **confluent**.

Extended Flow Diagrams III

By Lemma 43 and the Newman-Garbage Lemma, it suffices to show local confluent modulo garbage. Consider the critical pairs of the system. It turns out there are ten critical pairs, all but one of which are strongly joinable.



Thus, we do not have confluence, however by Lemma 44, the non-joinable critical pair is garbage, so by the Non-Garbage Critical Pair Lemma, we have local confluence modulo garbage, as required.