Introduction to Computable Analysis

Lecture 3: The Cantor Space II

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Standard Representations I

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In order to produce Type-2 generalizations of these theorems, we must first introduce general notations and representations, "naming systems".

Definition 1 (Notation and Representation)

1. A notation of a set M is a surjective function $\upsilon :\subseteq \Sigma^* \to M$.
2. A representation if a set M is a surjective function $\delta :\subseteq \Sigma^\omega \to M$.

Standard Representations II

Definition 2 (Name)

$p \in Y$ is a γ-name of $x \in M$ iff $\gamma :\subseteq Y \to M$ is a naming system and $\gamma(p) = x$.

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Definition 3 (Orders and Equivalences)

For $\gamma :\subseteq Y \to M$, $\gamma' :\subseteq Y' \to M'$ with $Y, Y' \in \{\Sigma^*, \Sigma^\omega\}$, we:

1. Say $f :\subseteq Y \to Y'$ reduces γ to γ' iff $\gamma(y) = (\gamma' f)(y)$ for all $y \in \mathrm{dom}(\gamma)$. Write $\gamma \leq \gamma'$ ($\gamma \leq_t \gamma'$) iff f is computable (continuous).
2. Write $\gamma \equiv \gamma'$ ($\gamma \equiv_t \gamma'$) iff $\gamma \leq \gamma' \leq \gamma$ ($\gamma \leq_t \gamma' \leq_t \gamma$).

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Clearly $\leq$, $\leq_t$ are preorders and $\equiv$, $\equiv_t$ are equivalences on the class of naming systems.

utm/smn-Properties Revisited

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If $\psi_w :\subseteq \Sigma^* \to \Sigma^*$ is the word function computed by the Turing Machine with code w, then the notation $w \to \psi_w$ of the computable word functions satisfy the utm/smn-properties.

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The function $(w, x) \mapsto \psi_w(x)$ is computable.

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Definition 5 ($\mathrm{smn}(\psi)$)

For every computable function $f :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$, there is a total computable function $r : \Sigma^* \to \Sigma^*$ with $r(x) \in \mathrm{dom}(\psi)$ for all $x \in \Sigma^*$ and $f(x, y) = \psi_{r(x)}(y)$ for all $x, y \in \Sigma^*$.

General utm/smn-Properties I

Let $a, b, c \in \{*, \omega\}$, G^{ab} be a set of functions, $g :\subseteq \Sigma^a \to \Sigma^b$, $\zeta :\subseteq \Sigma^c \to G^{ab}$ be a naming system of G^{ab}.

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Definition 6 ($\mathrm{utm}(\zeta)$)

There is a computable (universal) function $u : \Sigma^c \times \Sigma^a \to \Sigma^b$ with $\zeta_x(y) = u(x, y)$ for all $x \in \mathrm{dom}(\zeta)$ and $y \in \Sigma^a$.

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Definition 7 ($\mathrm{smn}(\zeta)$)

For every computable function $f :\subseteq \Sigma^c \times \Sigma^a \to \Sigma^b$, there is a total computable function $s : \Sigma^c \to \Sigma^c$ with $s(x) \in \mathrm{dom}(\zeta)$ for all $x \in \Sigma^c$ and $f(x, y) = \zeta_{s(x)}(y)$ for all $x \in \Sigma c$ and $y \in \Sigma^a$.

General utm/smn-Properties II

Definition 8 (Notation ξ^{ab})

Consider a canonical encoding of Type-2 machines with one input tape by words $w \in \Sigma^*$ such that the set TC of words is recursive. For all $a, b \in \{*, \omega\}$, define:

$$P^{ab} := \{f :\subseteq \Sigma^a \to \Sigma^b \mid f \text{ is computable}\}$$

and a notation $\xi^{ab} : \Sigma^* \to P^{ab}$ of the set P^{ab}, where $\xi^{ab}(w)$ is undefined on all non-code words, otherwise equals the function $f \in P^{ab}$ computed by the Type-2 machine with code w.

General utm/smn-Properties II

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Theorem 9

For all $a, b \in \{*, \omega\}$, we have $\mathrm{utm}(\xi^{ab})$ and $\mathrm{smn}(\xi^{ab})$.

Classes of Continuous Functions I

Definition 10

1. $F^{**} := \{f \mid f :\subseteq \Sigma^* \to \Sigma^*\}$;
2. $F^{*\omega} := \{f \mid f :\subseteq \Sigma^* \to \Sigma^\omega\}$;
3. $F^{\omega*} := \{f \mid f :\subseteq \Sigma^\omega \to \Sigma^*$ continuous and $\mathrm{dom}(f)$ open$\}$;
4. $F^{\omega\omega} := \{f \mid f :\subseteq \Sigma^\omega \to \Sigma^\omega$ continuous and $\mathrm{dom}(f)$ a G_δ set$\}$.

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Theorem 11

1. $F^{\omega *} = \{h_* \mid h :\subseteq \Sigma^* \to \Sigma^*$ *has prefix-free domain*$\}$;
2. $F^{\omega\omega} = \{h_\omega \mid h :\subseteq \Sigma^* \to \Sigma^*$ *is monotone*$\}$.

Classes of Continuous Functions II

Theorem 12

1. *Every continuous* $f :\subseteq \Sigma^\omega \to \Sigma^*$ *has an extension in* $F^{\omega *}$;
2. *Every continuous* $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ *has an extension in* $F^{\omega\omega}$.

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Functions from F^{ab} are "essentially" closed under composition.

Theorem 13

If $g \in F^{ab}$, $f \in F^{bc}$ *where* $a, b, c \in \{*, \omega\}$, *then:*

1. *If* $b = \omega$ *and* $c = *$, *then* $f \circ g$ *has an extension* $d \in F^{a*}$ *with* $\mathrm{dom}(d) \cap \mathrm{dom}(g) = \mathrm{f} \circ \mathrm{g}$;
2. *Otherwise,* $f \circ g \in F^{ac}$.

Classes of Continuous Functions III

Definition 14 (Standard Representation of F^{ab})

For all $a, b \in \{*, \omega\}$, define the standard representation $\eta^{ab} : \Sigma^\omega \to F^{ab}$ of F^{ab} by $\eta^{ab}(\langle x, p\rangle)(y) := \xi_x^{\omega b}\langle p, y\rangle$ for all $x \in \Sigma^*, p \in \Sigma^\omega, y \in \Sigma^a$, and $\eta^{ab}(q)(\_) := \bot$ if for no $x \in \Sigma^*$, $\iota(x) \sqsubseteq q$.

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Lemma 15

The representations η^{ab} are well-defined.

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Lemma 15

The representations η^{ab} are well-defined.

Computable functions have computable names.

Lemma 16

A function $f :\subseteq \Sigma^a \to \Sigma^b$ is computable iff $f = \eta_p^{ab}$ for some computable $p \in \Sigma^\omega$.

Important Theorems

Theorem 17

For all $a, b \in \{*, \omega\}$*, we have* $\mathrm{utm}(\eta^{ab})$ *and* $\mathrm{smn}(\eta^{ab})$*.*

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Theorem 18

For notations β, γ, δ *of* G^{ab} *with* $\mathrm{utm}(\delta)$ *and* $\mathrm{smn}(\delta)$*, we have:*

1. $(\beta \leq \gamma \wedge \mathrm{utm}(\gamma)) \Rightarrow \mathrm{utm}(\beta)$;
2. $(\mathrm{smn}(\beta) \wedge \beta \leq \gamma) \Rightarrow \mathrm{smn}(\gamma)$;
3. $(\mathrm{utm}(\beta) \wedge \mathrm{smn}(\gamma)) \Rightarrow \beta \leq \gamma$;
4. $\mathrm{utm}(\beta) \Leftrightarrow \beta \leq \delta$;
5. $\mathrm{smn}(\beta) \Leftrightarrow \delta \leq \beta$;
6. $(\mathrm{utm}(\gamma) \wedge \mathrm{smn}(\gamma)) \Leftrightarrow \gamma \equiv \delta$.

A Topological View

For a representation δ of F^{ab}, define:

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Definition 19 ($\mathrm{tutm}(\delta)$)

There is a $u \in F^{\omega b}$ such that $u\langle p, y\rangle = \delta_p(y)$ for all $p \in \mathrm{dom}(\delta)$ and $y \in \Sigma^a$.

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Definition 20 ($\mathrm{tsmn}(\delta)$)

For every $f \in F^{\omega b}$, there is a continuous $s : \Sigma^\omega \to \Sigma^\omega$ with $s(p) \in \mathrm{dom}$ and $f\langle p, y\rangle = \delta_{s(p)}(y)$ for all $p \in \Sigma^\omega$ and $y \in \Sigma^a$.

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Theorem 21

For all $a, b \in \{*, \omega\}$, we have $\mathrm{tutm}(\eta^{ab})$ and $\mathrm{tsmn}(\eta^{ab})$.

Decidable Sets I

Definition 22 (Type-2 Decidability)

Consider $X \subseteq Z \subseteq Y := Y_1 \times \cdots \times Y_k$ where $k \geq 1$ and $Y_1, \ldots, Y_k \in \{\Sigma^*, \Sigma^\omega\}$. Then:

1. X is called r.e. open in Z iff $X = \mathrm{dom}(f) \cap Z$ for some computable function $f :\subseteq Y \to \Sigma^\star$;
2. X is called decidable in Z iff both X and $Z \setminus X$ are r.e. open in Z;
3. X is called open in Z iff $X = U \cap Z$ for some open set $U \subseteq Y$;
4. X is called closed in Z iff $Z \setminus X$ is open in Z.

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3. X is called open in Z iff $X = U \cap Z$ for some open set $U \subseteq Y$;
4. X is called closed in Z iff $Z \setminus X$ is open in Z.

Lemma 23

X r.e. open (decidable) in Z implies X is open (clopen) in Z.

Decidable Sets II

Theorem 24

Let $X \subseteq Z \subseteq Y$ be as before. Then X is clopen (decidable) in Z iff there is a continuous (computable) function $f : Y \to \Sigma^*$ with $f(z) = 1$ if $z \in X$ and $f(z) = 0$ if $z \in Z \setminus X$.

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Let $X \subseteq Z \subseteq Y$ be as before. Then X is clopen (decidable) in Z iff there is a continuous (computable) function $f : Y \to \Sigma^*$ with $f(z) = 1$ if $z \in X$ and $f(z) = 0$ if $z \in Z \setminus X$.

Recall that each decidable set $X \subseteq \Sigma^\omega$ has the form $X = A\Sigma^\omega$ for some finite $A \subseteq \Sigma^*$.

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Clearly, Z is decidable in X.

Example 26

If X is r.e. open (decidable), then X is r.e. open (decidable) in Z whenever $X \subseteq Z$.

Decidable Sets II

Example 27

The set $X = \{p \in \Sigma^\omega \mid p \neq o^\omega\}$ is r.e. open but its complement is not, since it does not contain a set $w\Sigma^\omega$. Thus, X is not closed, so not clopen, so not decidable.

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Example 28

$X \subseteq Y_1 \times \cdots \times Y_k$ is r.e. open (decidable) iff $\langle X \rangle$ is r.e. open (decidable) in $\langle Y_1 \times \cdots \times Y_k \rangle$.

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Example 29

Union and intersection of r.e. open sets are r.e. open. For decidable sets, complement is decidable too.

Decidable Sets III

Example 30

If $f :\subseteq Y_1 \times \cdots \times Y_k \to Y_0$ is computable and $U \subseteq W \subseteq Y_0$ is r.e. open (decidable) in W, then $f^{-1}[U]$ is r.e. open (decidable) in $f^{-1}[W]$.

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Theorem 31

For $X \subseteq Y := Y_1 \times \cdots Y_k$ ($Y_1, \ldots Y_k \in \{\Sigma^*, \Sigma^\omega\}$), the following properties are equivalent:

1. *X is r.e. open;*
2. *$X = A \circ Y$ for some r.e. $A \subseteq (\Sigma^*)^k$;*
3. *X is open and $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is r.e. open.*

Recursive Open Sets

Definition 32

For any open $X \subseteq Y := Y_1 \times \cdots Y_k$, define:

1. X is r.e. iff $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is r.e.;
2. X is co-r.e. iff $\{y \in (\Sigma^*)^k \mid y \circ Y \not\subseteq X\}$ is r.e.;
3. X is recursive iff $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is decidable.

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Thus, an open set $X \subseteq Y$ is recursive iff it is both r.e. and co-r.e.. Notice that if $Y = (\Sigma^*)^k$, then X is (co-)r.e. (recursive) iff it is in the usual sense.

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Thus, an open set $X \subseteq Y$ is recursive iff it is both r.e. and co-r.e.. Notice that if $Y = (\Sigma^*)^k$, then X is (co-)r.e. (recursive) iff it is in the usual sense.

Recall that r.e. open subsets of Y are closed under both union and intersection. The recursive open sets are only closed under union. Note that while decidable implies recursive, the reverse is not necessarily true.