

Introduction to Computable Analysis

Lecture 3: The Cantor Space II

Graham Campbell

Summer 2019

Standard Representations I

In mathematical logic, a Gödel numbering is a function that assigns each symbol and well-formed formula to a unique natural number, used by K. Gödel for the proof of his incompleteness theorems in 1931.

Equivalently, a Gödel numbering $\varphi : \mathbb{N} \rightarrow P^{(1)}$ of the set $P^{(1)}$ of partial recursive functions (computable number functions) $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ is defined uniquely up to equivalence in Type-1 recursion theory. This is due to the utm-theorem and smn-theorem.

In order to produce Type-2 generalizations of these theorems, we must first introduce general notations and representations, “naming systems”.

Definition 1 (Notation and Representation)

- 1 A notation of a set M is a surjective function $\nu : \subseteq \Sigma^* \rightarrow M$.
- 2 A representation of a set M is a surjective function $\delta : \subseteq \Sigma^\omega \rightarrow M$.

Standard Representations II

Definition 2 (Name)

$p \in Y$ is a γ -name of $x \in M$ iff $\gamma : \subseteq Y \rightarrow M$ is a naming system and $\gamma(p) = x$.

Definition 3 (Orders and Equivalences)

For $\gamma : \subseteq Y \rightarrow M$, $\gamma' : \subseteq Y' \rightarrow M'$ with $Y, Y' \in \{\Sigma^*, \Sigma^\omega\}$, we:

- 1 Say $f : \subseteq Y \rightarrow Y'$ reduces γ to γ' iff $\gamma(y) = (\gamma'f)(y)$ for all $y \in \text{dom}(\gamma)$. Write $\gamma \leq \gamma'$ ($\gamma \leq_t \gamma'$) iff f is computable (continuous).
- 2 Write $\gamma \equiv \gamma'$ ($\gamma \equiv_t \gamma'$) iff $\gamma \leq \gamma' \leq \gamma$ ($\gamma \leq_t \gamma' \leq_t \gamma$).

Clearly \leq , \leq_t are preorders and \equiv , \equiv_t are equivalences on the class of naming systems.

utm/smn-Properties Revisited

In elementary computability theory, Turing Machines computing word functions $f : \subseteq \Sigma^* \rightarrow \Sigma^*$ are encoded canonically by words from Σ^* .

If $\psi_w : \subseteq \Sigma^* \rightarrow \Sigma^*$ is the word function computed by the Turing Machine with code w , then the notation $w \rightarrow \psi_w$ of the computable word functions satisfy the utm/smn-properties.

Definition 4 (utm(ψ))

The function $(w, x) \mapsto \psi_w(x)$ is computable.

Definition 5 (smn(ψ))

For every computable function $f : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$, there is a total computable function $r : \Sigma^* \rightarrow \Sigma^*$ with $r(x) \in \text{dom}(\psi)$ for all $x \in \Sigma^*$ and $f(x, y) = \psi_{r(x)}(y)$ for all $x, y \in \Sigma^*$.

General utm/smn-Properties I

Let $a, b, c \in \{*, \omega\}$, G^{ab} be a set of functions, $g : \subseteq \Sigma^a \rightarrow \Sigma^b$,
 $\zeta : \subseteq \Sigma^c \rightarrow G^{ab}$ be a naming system of G^{ab} .

Definition 6 (utm(ζ))

There is a computable (universal) function $u : \Sigma^c \times \Sigma^a \rightarrow \Sigma^b$ with
 $\zeta_x(y) = u(x, y)$ for all $x \in \text{dom}(\zeta)$ and $y \in \Sigma^a$.

Definition 7 (smn(ζ))

For every computable function $f : \subseteq \Sigma^c \times \Sigma^a \rightarrow \Sigma^b$, there is a total
 computable function $s : \Sigma^c \rightarrow \Sigma^c$ with $s(x) \in \text{dom}(\zeta)$ for all $x \in \Sigma^c$ and
 $f(x, y) = \zeta_{s(x)}(y)$ for all $x \in \Sigma^c$ and $y \in \Sigma^a$.

General utm/smn-Properties II

Definition 8 (Notation ξ^{ab})

Consider a canonical encoding of Type-2 machines with one input tape by words $w \in \Sigma^*$ such that the set TC of words is recursive. For all $a, b \in \{*, \omega\}$, define:

$$P^{ab} := \{f : \subseteq \Sigma^a \rightarrow \Sigma^b \mid f \text{ is computable}\}$$

and a notation $\xi^{ab} : \Sigma^* \rightarrow P^{ab}$ of the set P^{ab} , where $\xi^{ab}(w)$ is undefined on all non-code words, otherwise equals the function $f \in P^{ab}$ computed by the Type-2 machine with code w .

Theorem 9

For all $a, b \in \{, \omega\}$, we have $\text{utm}(\xi^{ab})$ and $\text{smn}(\xi^{ab})$.*

Classes of Continuous Functions I

Definition 10

- 1 $F^{**} := \{f \mid f : \subseteq \Sigma^* \rightarrow \Sigma^*\};$
- 2 $F^{*\omega} := \{f \mid f : \subseteq \Sigma^* \rightarrow \Sigma^\omega\};$
- 3 $F^{\omega*} := \{f \mid f : \subseteq \Sigma^\omega \rightarrow \Sigma^* \text{ continuous and } \text{dom}(f) \text{ open}\};$
- 4 $F^{\omega\omega} := \{f \mid f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \text{ continuous and } \text{dom}(f) \text{ a } G_\delta \text{ set}\}.$

Theorem 11

- 1 $F^{\omega*} = \{h_* \mid h : \subseteq \Sigma^* \rightarrow \Sigma^* \text{ has prefix-free domain}\};$
- 2 $F^{\omega\omega} = \{h_\omega \mid h : \subseteq \Sigma^* \rightarrow \Sigma^* \text{ is monotone}\}.$

Classes of Continuous Functions II

Theorem 12

- 1 Every continuous $f : \subseteq \Sigma^\omega \rightarrow \Sigma^*$ has an extension in $F^{\omega*}$;
- 2 Every continuous $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ has an extension in $F^{\omega\omega}$.

Functions from F^{ab} are “essentially” closed under composition.

Theorem 13

If $g \in F^{ab}$, $f \in F^{bc}$ where $a, b, c \in \{*, \omega\}$, then:

- 1 If $b = \omega$ and $c = *$, then $f \circ g$ has an extension $d \in F^{a*}$ with $\text{dom}(d) \cap \text{dom}(g) = \text{dom}(f \circ g)$;
- 2 Otherwise, $f \circ g \in F^{ac}$.

Classes of Continuous Functions III

Definition 14 (Standard Representation of F^{ab})

For all $a, b \in \{*, \omega\}$, define the standard representation $\eta^{ab} : \Sigma^\omega \rightarrow F^{ab}$ of F^{ab} by $\eta^{ab}(\langle x, p \rangle)(y) := \xi_x^{\omega b} \langle p, y \rangle$ for all $x \in \Sigma^*$, $p \in \Sigma^\omega$, $y \in \Sigma^a$, and $\eta^{ab}(q)(-) := \perp$ if for no $x \in \Sigma^*$, $\iota(x) \sqsubseteq q$.

Lemma 15

The representations η^{ab} are well-defined.

Computable functions have computable names.

Lemma 16

A function $f : \subseteq \Sigma^a \rightarrow \Sigma^b$ is computable iff $f = \eta_p^{ab}$ for some computable $p \in \Sigma^\omega$.

Important Theorems

Theorem 17

For all $a, b \in \{, \omega\}$, we have $\text{utm}(\eta^{ab})$ and $\text{smn}(\eta^{ab})$.*

Theorem 18

For notations β, γ, δ of G^{ab} with $\text{utm}(\delta)$ and $\text{smn}(\delta)$, we have:

- 1 $(\beta \leq \gamma \wedge \text{utm}(\gamma)) \Rightarrow \text{utm}(\beta)$;
- 2 $(\text{smn}(\beta) \wedge \beta \leq \gamma) \Rightarrow \text{smn}(\gamma)$;
- 3 $(\text{utm}(\beta) \wedge \text{smn}(\gamma)) \Rightarrow \beta \leq \gamma$;
- 4 $\text{utm}(\beta) \Leftrightarrow \beta \leq \delta$;
- 5 $\text{smn}(\beta) \Leftrightarrow \delta \leq \beta$;
- 6 $(\text{utm}(\gamma) \wedge \text{smn}(\gamma)) \Leftrightarrow \gamma \equiv \delta$.

A Topological View

For a representation δ of F^{ab} , define:

Definition 19 ($\text{tutm}(\delta)$)

There is a $u \in F^{\omega b}$ such that $u\langle p, y \rangle = \delta_p(y)$ for all $p \in \text{dom}(\delta)$ and $y \in \Sigma^a$.

Definition 20 ($\text{tsmn}(\delta)$)

For every $f \in F^{\omega b}$, there is a continuous $s : \Sigma^\omega \rightarrow \Sigma^\omega$ with $s(p) \in \text{dom}$ and $f\langle p, y \rangle = \delta_{s(p)}(y)$ for all $p \in \Sigma^\omega$ and $y \in \Sigma^a$.

Theorem 21

For all $a, b \in \{, \omega\}$, we have $\text{tutm}(\eta^{ab})$ and $\text{tsmn}(\eta^{ab})$.*

Decidable Sets I

Definition 22 (Type-2 Decidability)

Consider $X \subseteq Z \subseteq Y := Y_1 \times \cdots \times Y_k$ where $k \geq 1$ and $Y_1, \dots, Y_k \in \{\Sigma^*, \Sigma^\omega\}$. Then:

- 1 X is called r.e. open in Z iff $X = \text{dom}(f) \cap Z$ for some computable function $f : \subseteq Y \rightarrow \Sigma^*$;
- 2 X is called decidable in Z iff both X and $Z \setminus X$ are r.e. open in Z ;
- 3 X is called open in Z iff $X = U \cap Z$ for some open set $U \subseteq Y$;
- 4 X is called closed in Z iff $Z \setminus X$ is open in Z .

Lemma 23

X r.e. open (decidable) in Z implies X is open (clopen) in Z .

Decidable Sets II

Theorem 24

Let $X \subseteq Z \subseteq Y$ be as before. Then X is clopen (decidable) in Z iff there is a continuous (computable) function $f : Y \rightarrow \Sigma^*$ with $f(z) = 1$ if $z \in X$ and $f(z) = 0$ if $z \in Z \setminus X$.

Recall that each decidable set $X \subseteq \Sigma^\omega$ has the form $X = A\Sigma^\omega$ for some finite $A \subseteq \Sigma^*$.

Example 25

Clearly, Z is decidable in X .

Example 26

If X is r.e. open (decidable), then X is r.e. open (decidable) in Z whenever $X \subseteq Z$.

Decidable Sets II

Example 27

The set $X = \{p \in \Sigma^\omega \mid p \neq o^\omega\}$ is r.e. open but its complement is not, since it does not contain a set $w\Sigma^\omega$. Thus, X is not closed, so not clopen, so not decidable.

Example 28

$X \subseteq Y_1 \times \cdots \times Y_k$ is r.e. open (decidable) iff $\langle X \rangle$ is r.e. open (decidable) in $\langle Y_1 \times \cdots \times Y_k \rangle$.

Example 29

Union and intersection of r.e. open sets are r.e. open. For decidable sets, complement is decidable too.

Decidable Sets III

Example 30

If $f : \subseteq Y_1 \times \cdots \times Y_k \rightarrow Y_0$ is computable and $U \subseteq W \subseteq Y_0$ is r.e. open (decidable) in W , then $f^{-1}[U]$ is r.e. open (decidable) in $f^{-1}[W]$.

Theorem 31

For $X \subseteq Y := Y_1 \times \cdots \times Y_k$ ($Y_1, \dots, Y_k \in \{\Sigma^*, \Sigma^\omega\}$), the following properties are equivalent:

- 1 X is r.e. open;
- 2 $X = A \circ Y$ for some r.e. $A \subseteq (\Sigma^*)^k$;
- 3 X is open and $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is r.e. open.

Recursive Open Sets

Definition 32

For any open $X \subseteq Y := Y_1 \times \cdots \times Y_k$, define:

- 1 X is r.e. iff $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is r.e.;
- 2 X is co-r.e. iff $\{y \in (\Sigma^*)^k \mid y \circ Y \not\subseteq X\}$ is r.e.;
- 3 X is recursive iff $\{y \in (\Sigma^*)^k \mid y \circ Y \subseteq X\}$ is decidable.

Thus, an open set $X \subseteq Y$ is recursive iff it is both r.e. and co-r.e..

Notice that if $Y = (\Sigma^*)^k$, then X is (co-)r.e. (recursive) iff it is in the usual sense.

Recall that r.e. open subsets of Y are closed under both union and intersection. The recursive open sets are only closed under union. Note that while decidable implies recursive, the reverse is not necessarily true.