# Map Diagrams of $\mathcal{T}_X$: Graph Grammars

Graham Campbell

October 27, 2018

**Definition 1.** A **semigroup** is some non-empty set $S$ with an associative binary operation on $S$.

**Proposition 2.** The **full transformation semigroup** $\mathcal{T}_X$ on any set $X$ is the set of all functions $X \to X$, with operation composition of maps.

**Definition 3.** If $S$ is a semigroup, then $a \in S$ is **idempotent** iff $a^2 = a$. We denote by $E(S)$, the set of idempotent elements of $S$.

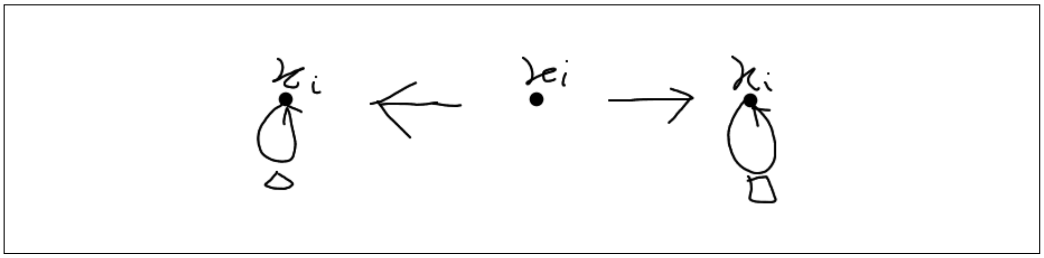
**Lemma 4.** An element $\mathcal{T}_X$ is idempotent exactly when its restriction to its image is the identity map. That is $\epsilon \in E(\mathcal{T}_X)$ iff $\epsilon\big|_{X\epsilon} = I_{X\epsilon}$.

Using the above lemma, we can precisely characterise the map diagrams of the idempotents of $\mathcal{T}_X$ using a graph grammar.

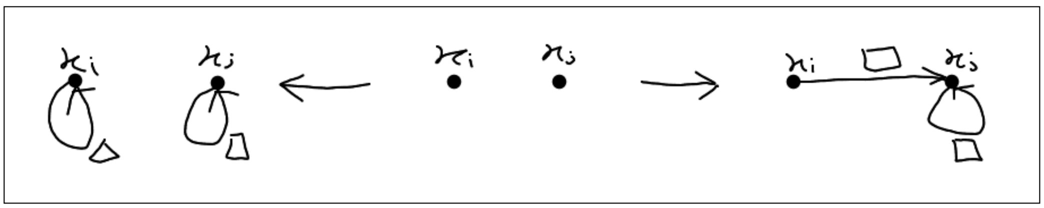
**Definition 5.** For any finite set $X$, let $\mathcal{G}(X) = (\mathcal{L}, \mathcal{N}, \mathcal{R}, S)$ be a graph grammar, and set $n = |X|$. With $\square$ denoting the empty label, we define:

1. $\mathcal{L} = (\mathcal{L}_V = X, \mathcal{L}_E = \{\square, \triangle\})$;
2. $\mathcal{N} = (\mathcal{N}_V = \emptyset, \mathcal{N}_E = \{\triangle\})$;
3. $\mathcal{R} = \{r_1, ..., r_n, r_{n+1}, ..., r_{n+n^2}\}$;
4. $S$ is the following graph, where the $x_i$ are all the elements from $X$:

We define rules $r_1, ..., r_n$ for all $x_i \in X$ by:



We define rules $r_{n+1}, ..., r_{n+n^2}$ for all $x_i, x_j \in X$ by:



**Proposition 6.** $L(\mathcal{G})$ is exactly the set of all map diagrams of the elements of $E(\mathcal{T}_X)$. Since map diagrams characterize the function exactly, then the language of the grammar is computing the idempotent elements for us!

**Remark 7.** The derivations that are terminally labelled are exactly those of length $n$, and the system is terminating, since those terminally labelled derivations are of maximal length.