

Categories & Algebras 3

Automata and Monoids II

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Syntactic Congruence

Let L be a language over A .

Definition (Context)

For $u \in A^*$, we define the context of u :

$$C_L(u) = \{(w, z) \in A^* \times A^* \mid wuz \in L\}.$$

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Definition (Congruence)

We define the syntactic congruence of L :

$$u \sim_L v \text{ iff } C_L(u) = C_L(v).$$

Clearly, \sim_L is an equivalence on A^* .

Syntactic Monoid

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If $u \sim_L v$ and $u' \sim_L v'$, then $uu' \sim_L vv'$.

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$$M(L) = \{[w] \mid w \in A^*\},$$

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and define a product on $M(L)$ by $[u][u'] = [uu']$.

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This product is a well defined binary operation on $M(L)$, and $M(L)$ is a monoid under it.

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So, there are three \sim_L -classes:

- 1 $[\lambda] = \{\lambda\} = 1$,
- 2 $[a] = \{a, b\} = L$,
- 3 $[a^2] = \{w \in A^* \mid |w| > 1\} = T$.

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In fact, $C_L(w) = \emptyset$ for all $|w| \geq 3$.

Monoid Morphisms

Definition

Let M, N be monoids. Then map $\theta : M \rightarrow N$ is a morphism iff:

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Example

Let $\theta : A^* \rightarrow \mathbb{N}^0$ be $w\theta = |w|$.

Then $\lambda\theta = |\lambda| = 0$, and for all $v, w \in A^*$:

$$(vw)\theta = |vw| = |v| + |w| = v\theta + w\theta.$$

Thus, θ is a monoid morphism.

Free Monoids

Theorem

Let A be an alphabet, M a monoid, and $\varphi : A \rightarrow M$ a function.

Then, there exists a unique morphism $\theta : A^ \rightarrow M$ s.t.*

$$\forall a \in A, a\theta = a\varphi.$$

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More generally, any monoid is called free if it is isomorphic to A^* for some set A . For example \mathbb{N}^0 is a free monoid.

Recognition by Monoid

Definition (Recognition)

Let $L \subseteq A^*$ and M be a monoid.

Then, L is recognised by M iff there exists a morphism $\theta : A^* \rightarrow M$ such that $L = (L\theta)\theta^{-1}$.

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Lemma

Let L be a language over A .

Then, L is recognised by $M(L) = \{[w] \mid w \in A^\}$.*

Regular Languages

Theorem

The following are equivalent for a language L over A :

- 1 $M(L)$ is finite,
- 2 L is recognised by a finite monoid,
- 3 $L \in \text{Rec } A^*$.

We say such languages are regular.

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Corollary

Let $L = L(\mathcal{A})$ for minimal FSA \mathcal{A} .

Then, $M(L) \cong M(\mathcal{A})$.

Applications

We can use monoids to show closure properties under Boolean operations, by which I mean $\cup, \cap, ^c$.

Lemma

$L, K \in \text{Rec } A^* \Rightarrow L \cap K \in \text{Rec } A^*$.

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We can define a new class of languages using these operations.

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$L \subseteq A^*$ is star-free iff it is finite, or can be obtained by applying Boolean operations a finite number of times.

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Definition (Star-Free)

$L \subseteq A^*$ is star-free iff it is finite, or can be obtained by applying Boolean operations a finite number of times.

Lemma

If X is the set of all star-free languages over A , then $X \subset \text{Rat } A^$.*

Subgroups

Definition (Subgroup)

Let M be a monoid, and $G \subseteq M$.

Then, G is a subgroup of M iff:

- 1 $a, b \in G \Rightarrow ab \in G$,
- 2 $\exists e \in G, \forall a \in G, ea = ae$,
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Lemma

Let $E(M)$ denote the set of idempotent elements of monoid M .

Then, $e \in E(M) \Rightarrow \{e\}$ is a subgroup.

Schutzenberger's Theorem

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A language L is star-free iff:

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Example

Is $L = \{a^{2^n} \mid n \geq 0\}$ over alphabet $A = \{a, b\}$ star-free?

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Remark: In general, the problem of testing if an element behaves as an identity is actually undecidable! However, in the case of finite sets, we're fine. See the **Undecidable Problems** lecture.