Categories & Algebras 2

Automata and Monoids I

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Alphabets and Words

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Definition (Kleene operators)

We define $A^+ = \bigcup_{i \geq 1} A^i$ to be the set of all non-empty words over A, and $A^* = A^+ \cup \{\lambda\}$ where λ is the empty string.

Languages and Monoids

Definition (Language)

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All the string and languages operations we know from MFCS carry over. We are not going to recap them again.

Automata

Definition (FSA)

Recall from MFCS that a FSA is a 5-tuple:

$$\mathcal{A} = (A, Q, \delta, q_0, F),$$

where transition function $\delta : Q \times A \to Q$ can be extended in the obvious way to give $\delta : Q \times A^* \to Q$.

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where transition function $\delta : Q \times A \to Q$ can be extended in the obvious way to give $\delta : Q \times A^* \to Q$.

Definition (Language recognised)

The language recognised by $\mathcal{A}$ is:

$$L(\mathcal{A}) = \{w \in A^* \mid \delta(q_0, w) \in F\}.$$

Recognisable languages

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A language $L \subseteq A^*$ is recognisable iff there exists an FSA $\mathcal{A}$ such that:

$$L = L(\mathcal{A}),$$

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and we write $L \in \operatorname{Rec} A^*$.

We note that $A^*, \emptyset \in \operatorname{Rec} A^*$, and all singleton languages lie in $\operatorname{Rec} A^*$. This is easy to see by constructing automata.

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Lemmas 2

We remark that $\operatorname{Rec} A^*$ is not closed under infinite union or intersection, however, with finitely many applications, one can see:

1. L finite $\Rightarrow L \in \operatorname{Rec} A^*$,
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We will quote two more results without proof:

Lemma

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Rational Languages

Definition (Rational operations)

The rational operations on languages over A are:

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Definition (Rational languages)

$L \subseteq A^*$ is rational iff L is finite, or L can be obtained by applying rational operations a finite number of times.

We write $\mathrm{Rat}\, A^*$ for the set of rational languages over A.

Kleene's Theorem

From our earlier results, we know that $\operatorname{Rat} A^* \subseteq \operatorname{Rec} A^*$, however, we can take this one step further.

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Definition (Rational expressions)

A rational expression for a language L over A is one that expresses L using only finite languages and the rational operations a finite number of times.

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Definition (Submonoid)

Let M be a monoid, and $T \subseteq M$. Then T is a submonoid iff:

1. $1 \in T$,
2. $a, b \in T \Rightarrow ab \in T$,

and we write $T \leq M$.

Generators

Definition (Generator)

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Examples

1. $A^* = \langle A \rangle$ where A is an alphabet.
2. $\mathbb{N} = \langle P \rangle$ where P is the set of primes.

Transformation Monoids

Definition (Full transformation monoid)

If X is a non-empty set, then:

$$\mathcal{T}_X = \{\alpha \mid \alpha : X \to X\}$$

is a monoid under functional composition, with identity I_X, and is called the "full transformation monoid on X".

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Notation

From now onward, we will be using postfix notation for functional application, and if $X = \{1, 2, ...n\}$, write $\mathcal{T}_n$ for $\mathcal{T}_X$ and I_n for I_X.

It is also convenient to adopt two-row notation for elements of $\mathcal{T}_n$.

Constant Functions

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Remark

Note that:

1. $\forall \alpha \in \mathcal{T}_X, \alpha c_x = c_x$,
2. $\forall \alpha \in \mathcal{T}_X, c_x \alpha = c_{x\alpha}$.

Preliminary Result

Lemma

Let $\mathcal{A} = (A, Q, \delta, q_0, F)$ *be a FSA.*

For each $w \in A^*$, *let* $\sigma_w \in \mathcal{T}_Q$ *be defined by:*

$$q\sigma_w = \delta(q, w).$$

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Then, for any $w, v \in A^*$:

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Remark

In particular, we note that $q\sigma_\lambda = \delta(q, \lambda) = q = qI_Q$.

Transition Monoids

So, we have an identity σ_λ, and closed multiplication.

Definition (Transition monoid)

Thus, we have submonoid:

$$M(\mathcal{A}) = \{\sigma_w \mid w \in A^*\} \leq \mathcal{T}_Q.$$

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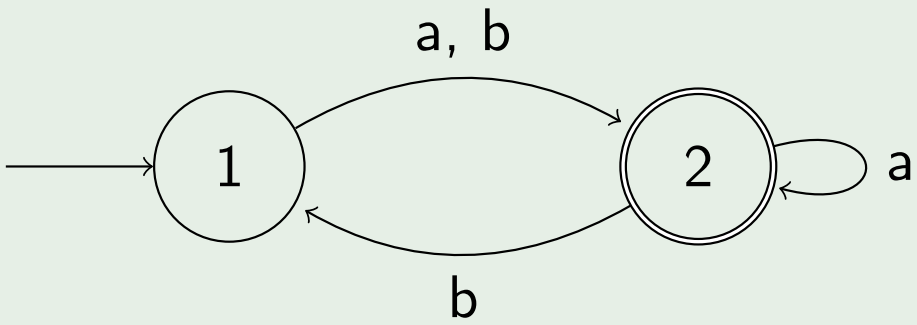
Lemma

$M(\mathcal{A}) = \langle \sigma_a \mid a \in A \rangle$, and is finite.

Example 1

Let $A = \{a, b\}$ and $Q = \{1, 2\}$.

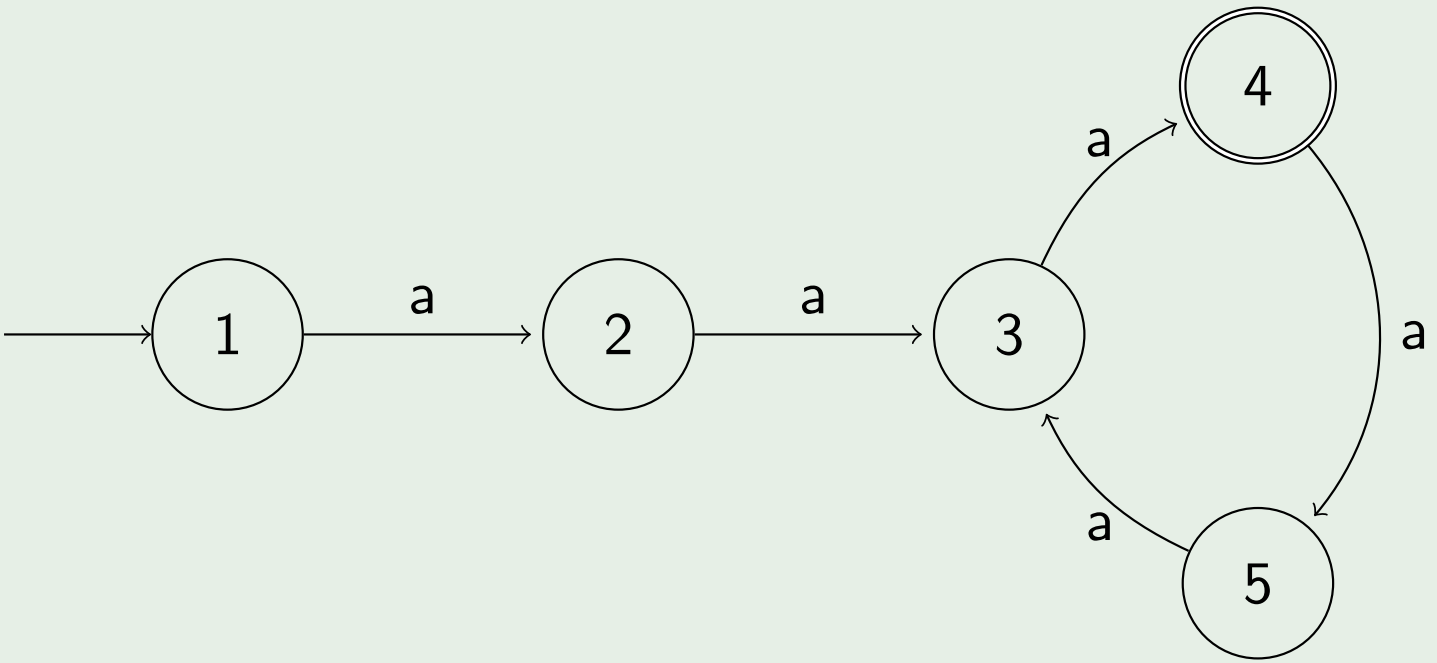
State Diagram



Example 2

Let $A = \{a\}$ and $Q = \{1, 2, 3, 4, 5\}$.

State Diagram



Example 3

Let $A = \{a, b\}$ and $Q = \{1, 2, 3\}$.

State Diagram